Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B4

REGRESSION MODELING OF GROUND-WATER FLOW

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Book 3
APPLICATIONS OF HYDRAULICS

7 Answers to Exercises

In this section answers to the exercises from the first six sections are given. For brevity, the exercises are not restated.

Problem 2.2-1

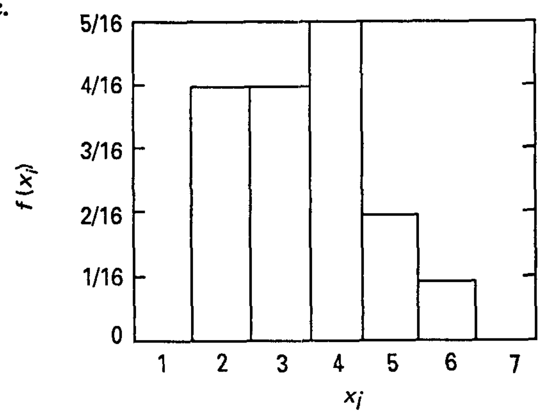
a. (RR),(RW),(RB),(WW),(WR),(WB),(BB),(BW),(BR).

b.

	R	W	B1	B2
R	RR	RW	RB1	RB2
W	WR	WW	WB1	WB2
B1	B1R	B1W	B1B1	B1B2
B2	B2R	B2W	B2B1	B2B2

P(RR)=1/16 P(WW)=1/16 P(BB)=1/4
P(RW)=1/16 P(WR)=1/16 P(BW)=1/8
P(RB)=1/8 P(WB)=1/8 P(BR)=1/8

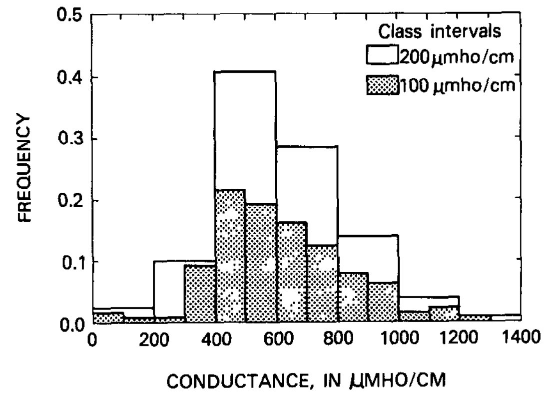
c.

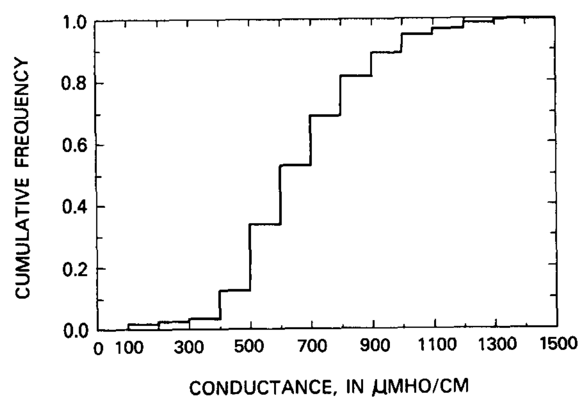


d. $P(X=4)=5/16$
(RR),(WB),(BW).

Problem 2.2-2

a. The effect of increasing the class interval size is to increase the frequency value for each class. This increase tends to make the histogram appear more peaked for larger intervals. Increasing the class interval also tends to smooth out irregularities.





b. $P(X \leq 600)=0.53$
$P(X>400)=0.88$
$P(400<X \leq 600)=0.41$
$P(X \leq 1300)=1.0.$

Problem 2.3-1

a. $\bar{x}=\sum_{(\text{all } i)} \bar{x}_i f_i^*=(2\times50+1\times150+1\times250+12\times350+28\times450+25\times550+21\times650+16\times750+10\times850+8\times950+2\times1050+3\times1150+1\times1250)/130$
$=612.$

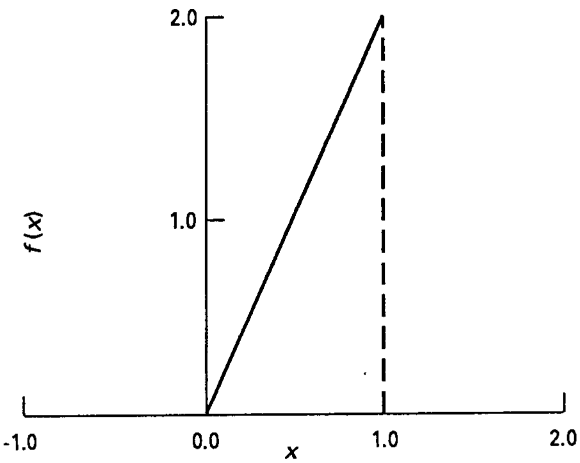
b. By replacing the integration in equation 2.3-3 with a summation, it is seen that the population mean μ_X for this random variable is represented by

$$\mu_X=\sum_{i=2}^{6} if_i$$
$$=(2\times1/4+3\times1/4+4\times5/16+5\times1/8+6\times1/16)$$
$$=3\frac{1}{2}\ .$$

Problem 2.3-2

a. Triangular density problem.

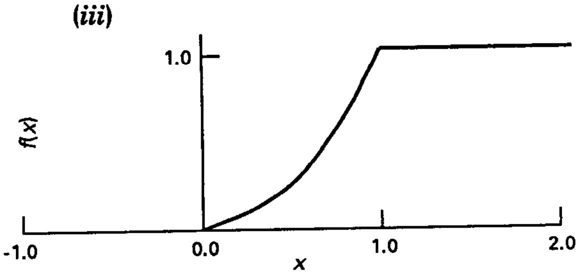
(*i*)



(*ii*)

$$F(x)=\int_{-\infty}^{x} f(s)ds=\begin{cases}1 & ,x>1\\ 2\int_0^x s\,ds=x^2 & ,0\le x\le 1\\ 0 & ,x<0\end{cases}$$

(*iii*)



$$E[X]=\int_{-\infty}^{\infty} sf(s)ds=2\int_0^1 s^2ds=\frac{2}{3}$$

$$E[X^2]=\int_{-\infty}^{\infty} s^2f(s)ds=2\int_0^1 s^3ds=\frac{1}{2}$$

$$\sigma_X^2=E[X^2]-(E[X])^2=\frac{1}{2}-\frac{4}{9}=\frac{1}{18}\ .$$

b. $\bar{x}=1.25\times0.051+1.75\times0.103+2.25\times0.103$
$+2.75\times0.154+3.25\times0.154+3.75\times0.128$
$+4.25\times0.128+4.75\times0.090+5.25\times0.026$
$+5.75\times0.064=3.36.$

$$s_x^{*2}=\sum_{i=1}^{10}(\bar{x}_i-\bar{x})^2f_i^*$$

$=(-2.11)^2\times0.051+(-1.61)^2\times0.103$
$+(-1.11)^2\times0.103+(-0.61)^2\times0.154$
$+(-0.11)^2\times0.154+(0.39)^2\times0.128$
$+(0.89)^2\times0.128+(1.39)^2\times0.090$
$+(1.89)^2\times0.026+(2.39)^2\times0.064=1.43$.

Problem 2.4-1

a. $P(X=3 \text{ and } Y=2)=P(X=3)P(Y=2)=1/36.$
b. $P(X+Y=5)=1/9.$
c. $P(Y=2|X=3)=1/6.$
d. $P(X+Y=5|X=3)=1/6.$
e. $P(X+Y\le5|X=3)=1/3.$

Problem 2.4-2

a. $E[Y]=a_1\mu_{X_1}+a_2\mu_{X_2}+a_3\mu_{X_3}$

$$\begin{aligned}\mathrm{Var}[Y]&=E\{[Y-E(Y)]^2\}=E\{[a_1(X_1-\mu_{X_1})\\&\quad+a_2(X_2-\mu_{X_2})+a_3(X_3-\mu_{X_3})]^2\}\\&=E[a_1^2(X_1-\mu_{X_1})^2+a_2^2(X_2-\mu_{X_2})^2\\&\quad+a_3^2(X_3-\mu_{X_3})^2\\&\quad+2a_1a_2(X_1-\mu_{X_1})(X_2-\mu_{X_2})\\&\quad+2a_1a_3(X_1-\mu_{X_1})(X_3-\mu_{X_3})\\&\quad+2a_2a_3(X_2-\mu_{X_2})(X_3-\mu_{X_3})]\\&=a_1^2\sigma_{X_1}^2+a_2^2\sigma_{X_2}^2+a_3^2\sigma_{X_3}^2\\&\quad+2a_1a_2\sigma_{X_1X_2}+2a_1a_3\sigma_{X_1X_3}\\&\quad+2a_2a_3\sigma_{X_2X_3}\end{aligned}$$

because $\sigma_{X_i}^2=E[(X_i-\mu_{X_i})^2]$ and $\sigma_{X_iX_j}=E[(X_i-\mu_{X_i})\cdot(X_j-\mu_{X_j})]$.

b.
$$\begin{aligned}\underline{a}\mathrm{Var}[\underline{X}]\,\underline{a}^T&=\underline{a}E[(\underline{X}-E[\underline{X}])(\underline{X}-E[\underline{X}])^T]\underline{a}^T\\&=\underline{a}E[\underline{X}\underline{X}^T-\underline{X}E[\underline{X}]^T-E[\underline{X}]\underline{X}^T\\&\quad+E[\underline{X}]E[\underline{X}]^T]\underline{a}^T\\&=\underline{a}(E[\underline{X}\underline{X}^T]-E[\underline{X}]E[\underline{X}]^T)\underline{a}^T\\&=E[\underline{a}\underline{X}\underline{X}^T\underline{a}^T]-E[\underline{a}\underline{X}]E[\underline{X}^T\underline{a}^T]\\&=\mathrm{Var}[\underline{a}\underline{X}]\end{aligned}$$

c.
$$\mathrm{Var}[\underline{Y}]=\begin{vmatrix}\sigma_{Y_1}^2 & \sigma_{Y_2Y_2} & \cdots & \sigma_{Y_1Y_p}\\ & \sigma_{Y_2}^2 & \cdots & \sigma_{Y_2Y_p}\\ \text{symmetry} & & & \vdots\\ & & & \sigma_{Y_p}^2\end{vmatrix}$$

$$\sigma^2_{Y_i}=\mathrm{Var}[Y_i]=\mathrm{Var}[\underline{a}_i\underline{X}]=\underline{a}_i\mathrm{Var}[\underline{X}]\underline{a}_i^T$$

$$\begin{aligned}\sigma_{Y_iY_j}&=E[Y_iY_j]-E[Y_i]E[Y_j]\\&=E[\underline{a}_i\underline{X}\underline{X}^T\underline{a}_j^T]-E[\underline{a}_i\underline{X}]E[\underline{X}^T\underline{a}_j^T]\\&=\underline{a}_i(E[\underline{X}\underline{X}^T]-E[\underline{X}]E[\underline{X}^T])\underline{a}_j^T\\&=\underline{a}_i\mathrm{Var}[\underline{X}]\underline{a}_j^T\end{aligned}$$

$$\begin{aligned}\mathrm{Var}[\underline{Y}]&=\begin{bmatrix}\underline{a}_1\\ \underline{a}_2\\ \vdots\\ \underline{a}_p\end{bmatrix}\mathrm{Var}[\underline{X}][\underline{a}_1^T,\underline{a}_2^T,\ldots,\underline{a}_p^T]\\&=\underline{A}\,\mathrm{Var}[\underline{X}]\underline{A}^T\end{aligned}$$

Problem 2.5-1

a. $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i=610$.

The results differ very little, indicating the validity of approximation in equation 2.3-1. The repeat values probably result from measurement errors, indicating that a value of the random variable could not be determined more precisely than one μmho/cm.

b. $\bar{x}=3.36 \qquad s_X^2=1.43$

$$\begin{aligned}\bar{t}_g&=10^{\bar{x}}=2{,}290 \text{ gal/d/ft}\\ d_g&=10^{(\bar{x}\pm s_X)}=\bar{t}_g\times10^{\pm s_X}\\&=(146,\ 35900)\text{ gal/d/ft.}\end{aligned}$$

The large dispersion strongly suggests that $\bar{t}_g$ may not represent the true geometric mean, since there is significant scatter in T in the vicinity of $\bar{t}_g$.

Problem 2.5-2

Let X be the specific conductance random variable and Y be the dissolved solids random variable. Then

$\sum x_i=14{,}999$	$\sum y_i=9{,}140$
$\sum x_i^2=11{,}743{,}379$	$\sum y_i^2=4{,}316{,}966$
$\sum x_iy_i=7{,}095{,}973$	$n=23$
$\bar{x}=652$	$\bar{y}=397$
$s_X^2=89{,}200$	$s_Y^2=31{,}300$
$s_X=299$	$s_Y=176$

$$r_{XY}=0.98 \ .$$

Problem 2.6-1

$$\bar{x}=3\times10^{-4} \qquad s_X^2=6.6\times10^{-5}$$

$$\begin{aligned}&P(\bar{X}^2/(S_X^2/n)\le F_\alpha(1,n-1))\\&=P(-\sqrt{F_\alpha(1,n-1)}\le\bar{X}/(S_X/\sqrt{n})\le\sqrt{F_\alpha(1,n-1)})\\&=P(-2.26\le\bar{X}/(S_X/\sqrt{n})\le2.26)=0.95\\&\bar{x}/(s_X/\sqrt{n})=0.12\ .\end{aligned}$$

The value of the statistic is well within the interval ($-2.26<0.12<2.26$). This result is expected, if the titration experiment is valid, as 95 percent of all values of the statistic $\bar{X}/(S_X/\sqrt{n})$, calculated from repeated random sampling, would be expected to fall in this interval. If the value had fallen outside the interval, one should feel uneasy because this should occur only 5 percent of the time. One would then be obligated to question whether the assumption $\mu_X=0$ inherent to the titration test is valid.

Problem 2.8-1

a. $\bar{x}=10 \qquad s_X^2=0.08 \qquad n=7$

$$P\left(\frac{(\bar{X}-\mu_X)^2}{S_X^2/n}\le F_\alpha(1,6)\right)$$

$$=P\left(-\sqrt{F_\alpha(1,6)}\le\frac{\bar{X}-\mu_X}{S_X/\sqrt{n}}\le\sqrt{F_\alpha(1,6)}\right)\ .$$

Therefore, an interval can be constructed from

$$-\sqrt{F_\alpha(1,6)}\le\frac{\bar{x}-\mu_X}{s_X/\sqrt{n}}\le\sqrt{F_\alpha(1,6)}$$

where $F_\alpha(1,6)=5.99$. Thus

$$-s_X\sqrt{\frac{F_\alpha(1,6)}{n}}\le(\bar{x}-\mu_X)\le s_X\sqrt{\frac{F_\alpha(1,6)}{n}}$$

or

$$\bar{x}+s_X\sqrt{\frac{F_\alpha(1,6)}{n}}\ge\mu_X\ge\bar{x}-s_X\sqrt{\frac{F_\alpha(1,6)}{n}}$$

and

$$10.26\ge\mu_X\ge9.74\ .$$

b. As 95 percent of all intervals so constructed will contain μ_X, there is a 0.95 probability that this interval contains μ_X.

Problem 2.9-1

a. $H_0{:}\mu_X=0$

$H_1{:}\mu_X\neq 0$.

b. From equation 2.6-21, the statement $P(\text{reject } H_0/H_0 \text{ true})=\alpha$ becomes

$$P\left(\frac{S_1/\sigma_1^2}{S_2^2/\sigma_2^2}>a \,\middle|\, \frac{\sigma_1^2}{\sigma_2^2}=1\right)=P(S_1^2/S_2^2>c)=0.05$$

where S_1^2/S_2^2 is an $F(24,15)$ random variable. Because the critical region is defined by values of $F(24,15)$ greater than or equal to c, c must be equal to $F_{0.05}(24,15)$ (see equation 2.6-15). Thus, $c=2.29$ and, because $s_1^2/s_2^2=1.31$, we accept H_0 at 0.05 significance level.

c. Hypothesis to be tested for rejection:

$$H_0{:}\mu_X=9.8 \ .$$

Alternate hypothesis:

$$H_1{:}\mu_X\neq 9.8 \ .$$

From equation 2.9-7,

$$P\left(\frac{\bar{X}-\mu_0}{S_X/\sqrt{n}}<-\sqrt{F_\alpha(\nu_1,\nu_2)}\right)$$

$$+P\left(\frac{\bar{X}-\mu_0}{S_X/\sqrt{n}}>\sqrt{F_\alpha(\nu_1,\nu_2)}\right)$$

$$=P\left(\left(\frac{\bar{X}-\mu_0}{S_X/\sqrt{n}}\right)^2>F_\alpha(1,n-1)\right)=\alpha \ .$$

Thus, the critical value for the statistic $(\bar{X}-\mu_0)^2/(S_X^2/n)$ is $F_\alpha(1,n-1)$, which, at the 0.05 significance level, has a value of 5.99. Because the statistic for the random sample in question takes on the value 3.50, we are forced to accept the possibility that $\mu_X=0$: We cannot safely reject H_0 at the 0.05 significance level.

Problem 3.1-1

Upon substitution of equation 4 into equation 3, one obtains

$$h=\beta_1X_1+\beta_2X_2+\beta_3X_3 \ .$$

For n observations

$$\underline{Y}=\underline{X}\underline{\beta}+\underline{\epsilon}$$

where $\underline{Y}-\underline{h}=\underline{\epsilon}$,

$$\underline{Y}=\begin{bmatrix}Y_1\\ \vdots\\ Y_{n_s}\end{bmatrix} \quad \underline{h}=\begin{bmatrix}h_1\\ \vdots\\ h_{n_s}\end{bmatrix} \quad \underline{\beta}=\begin{bmatrix}\beta_1\\ \vdots\\ \beta_3\end{bmatrix} \quad \underline{\epsilon}=\begin{bmatrix}\epsilon_1\\ \vdots\\ \epsilon_{n_s}\end{bmatrix}$$

and

$$\underline{X}=\begin{bmatrix}X_{11} & X_{12} & X_{13}\\ \vdots & \vdots & \vdots\\ X_{n_s1} & X_{n_s2} & X_{n_s3}\end{bmatrix} \ .$$

In the regression model:

$\underline{Y}$=observed dependent variable vector;
$\underline{h}$=computed dependent variable vector;
s=independent variable (distance along stream tube);
$\underline{X}$=sensitivities; and
$\underline{\beta}$=parameters.

a. $S(\underline{b})=[\underline{Y}-\underline{X}\underline{b}]^T[\underline{Y}-\underline{X}\underline{b}]$

where

$$\underline{b}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$$

That is,

$$S(\underline{b})=\left[Y_1-(b_1X_{11}+b_2X_{12}+b_3X_{13})\right]^2$$

$$+\left[Y_2-(b_1X_{21}+b_2X_{22}+b_3X_{23})\right]^2$$

$$+\ldots$$

$$+\left[Y_{n_s}-(b_1X_{n_s1}+b_2X_{n_s2}+b_3X_{n_s3})\right]^2.$$

b. $S(\underline{b})=[\underline{Y}-\underline{X}\underline{b}]^T\underline{\omega}[\underline{Y}-\underline{X}\underline{b}]$

where $\underline{\omega}=\underline{V}^{-1}$ and

$$\underline{V}=\begin{bmatrix} \sigma_1^2/\sigma^2 & & \underline{0} \\ & \sigma_2^2/\sigma^2 \ddots & \\ \underline{0} & & \sigma_{n_s}^2/\sigma^2 \end{bmatrix}$$

so that

$$\underline{V}^{-1}=\begin{bmatrix} \sigma_1^{-2} & & \underline{0} \\ & \sigma_2^{-2} \ddots & \\ \underline{0} & & \sigma_{n_s}^{-2} \end{bmatrix}\sigma^2 .$$

That is,

$$S(\underline{b})=\sigma^2\left\{\left[\frac{1}{\sigma_1}(Y_1-(b_1X_{11}+b_2X_{12}+b_3X_{13}))\right]^2 + \left[\frac{1}{\sigma_2}(Y_2-(b_1X_{21}+b_2X_{22}+b_3X_{23}))\right]^2 + \ldots + \left[\frac{1}{\sigma_{n_s}}(Y_{n_s}-(b_1X_{n_s1}+b_2X_{n_s2}+b_3X_{n_s3}))\right]^2\right\}.$$

c. $S(\underline{b})=[\underline{Y}-\underline{X}\underline{b}]^T\underline{\omega}[\underline{Y}-\underline{X}\underline{b}]$

where

$$\underline{Y}-\underline{X}\underline{b}=\begin{bmatrix} Y_1-(b_1X_{11} & +b_2X_{12} & +b_3X_{13}) \\ & \vdots & \\ Y_{n_s}-(b_1X_{n_s1} & +b_2X_{n_s2} & +b_3X_{n_s3}) \\ h_b & -b_2 & \end{bmatrix}$$

and $\underline{\omega}=\underline{V}^{-1}$, so that

$$\underline{V}=\begin{bmatrix} 1 & & \underline{0} \\ & \ddots\ 1 & \\ \underline{0} & & \sigma_{h_b}^2/\sigma^2 \end{bmatrix}$$

and

$$\underline{V}^{-1}=\begin{bmatrix} 1 & & \underline{0} \\ & \ddots\ 1 & \\ \underline{0} & & \sigma^2/\sigma_{h_b}^2 \end{bmatrix}$$

That is,

$$S(\underline{b})=\left[Y_1-(b_1X_{11}+b_2X_{12}+b_3X_{13})\right]^2 + \ldots + \left[Y_{n_s}-(b_1X_{n_s1}+b_2X_{n_s2}+b_3X_{n_s3})\right]^2 + \sigma^2\left[h_b-b_2\right]^2/\sigma_{h_b}^2 .$$

Problem 3.2-1

a. Note that if $\underline{\omega}$ is diagonal

$$\underline{X}^T\underline{\omega}\underline{X}=\left\{\sum_{i=1}^{n_s} X_{ik}\omega_iX_{ij}+\delta_{k2}\delta_{2j}\sigma^2/\sigma_{h_b}^2\right\} =\left\{\sum_{i=1}^{n_s+1} X_{ik}\omega_iX_{ij}\right\}$$

and

$$\underline{X}^T\underline{\omega}\underline{Y}=\left\{\sum_{i=1}^{n_s} X_{ik}\omega_iY_i+\delta_{k2}h_b\sigma^2/\sigma_{h_b}^2\right\} =\left\{\sum_{i=1}^{n_s+1} X_{ik}\omega_iY_i\right\}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

$$\underline{X}^T=\begin{bmatrix} X_{11} & X_{21} & & X_{n_s1} & 0 \\ X_{12} & X_{22} & \cdots & X_{n_s2} & 1 \\ X_{13} & X_{23} & & X_{n_s3} & 0 \end{bmatrix} =\begin{bmatrix} X_{11} & X_{21} & & X_{n_s1} & X_{n_s+1,1} \\ X_{12} & X_{22} & \cdots & X_{n_s2} & X_{n_s+1,2} \\ X_{13} & X_{23} & & X_{n_s3} & X_{n_s+1,3} \end{bmatrix}$$

$$\underline{\omega}=\begin{bmatrix} 1 & & \underline{0} \\ & \ddots\ 1 & \\ \underline{0} & & \sigma^2/\sigma_{h_b}^2 \end{bmatrix} =\begin{bmatrix} \omega_1 & & & \underline{0} \\ & \ddots & & \\ & & \omega_{n_s} & \\ \underline{0} & & & \omega_{n_s+1} \end{bmatrix}$$

and

$$\underline{Y}=\begin{bmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_{n_s} \\ h_b \end{bmatrix}=\begin{bmatrix} Y_1 \\ Y_2 \\ \cdots \\ Y_{n_s} \\ Y_{n_s+1} \end{bmatrix} .$$

Thus, the normal equations

$$\underline{X}^T\underline{\omega}\underline{X}\underline{\hat{b}}=\underline{X}^T\underline{\omega}\underline{Y}$$

can be written

$$\sum_{j=1}^{p}\sum_{i=1}^{n_s+1} X_{ik}\omega_i X_{ij}\hat{b}_j=\sum_{i=1}^{n_s+1} X_{ik}\omega_i Y_i , \qquad k=1,2,\ldots,p$$

or

$$\sum_{i=1}^{n_s+1}\left(\sum_{j=1}^{p} X_{ik}\omega_i X_{ij}\hat{b}_j\right)=\sum_{i=1}^{n_s+1} X_{ik}\omega_i Y_i ,\ k=1,2,\ldots,p.$$

As a comparison, the normal equations may be derived without using matrix techniques. Let $n=n_s+1$ so that

$$S(\underline{b})=\sum_{i=1}^{n} e_i^2\omega_i .$$

The model equations are given as

$$Y_i=\sum_{j=1}^{p} X_{ij}b_j+e_i , \qquad i=1,2,\ldots,n .$$

Therefore

$$S(\underline{b})=\sum_{i=1}^{n}\left(Y_i-\sum_{j=1}^{p} X_{ij}b_j\right)^2\omega_i .$$

Take the derivative with respect to any parameter b_k ($k=1,2,\ldots,p$):

$$\frac{\partial S}{\partial b_k}=\frac{\partial}{\partial b_k}\left[\sum_{i=1}^{n}\left(Y_i-\sum_{j=1}^{p} X_{ij}b_j\right)^2\omega_i\right] .$$

Because the derivative of a sum is the sum of derivatives, look at one term:

$$\frac{\partial}{\partial b_k}\left[\left(Y_i-\sum_{j=1}^{p} X_{ij}b_j\right)^2\omega_i\right]$$

$$=-2\left(Y_i-\sum_{j=1}^{p} X_{ij}b_j\right)\omega_i X_{ik} .$$

Thus,

$$\frac{\partial S}{\partial b_k}=-2\sum_{i=1}^{n}\left(Y_i-\sum_{j=1}^{p} X_{ij}b_j\right)\omega_i X_{ik}$$

Set $\frac{\partial S}{\partial b_k}=0$ to find the minimum so that

$$\sum_{i=1}^{n}\left(Y_i-\sum_{j=1}^{p} X_{ij}\hat{b}_j\right)\omega_i X_{ik}=0,\ k=1,2,\ldots,p$$

or

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{p} X_{ik}\omega_i X_{ij}\hat{b}_j\right)=\sum_{i=1}^{n} X_{ik}\omega_i Y_i ,\ k=1,2,\ldots,p.$$

b. Consider matrix $\underline{A}$ such that

$$\underline{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} .$$

Assume $\underline{A}$ is symmetric; its cofactor matrix $\underline{A}_c$ is

$$\underline{A}_c=\begin{bmatrix} (a_{22}a_{33}-a_{23})^2 & (a_{13}a_{23}-a_{12}a_{33}) & (a_{12}a_{23}-a_{13}a_{22}) \\ (a_{13}a_{23}-a_{12}a_{33}) & (a_{11}a_{33}-a_{13})^2 & (a_{12}a_{13}-a_{11}a_{23}) \\ (a_{12}a_{23}-a_{13}a_{22}) & (a_{12}a_{13}-a_{11}a_{23}) & (a_{11}a_{22}-a_{12})^2 \end{bmatrix} .$$

The determinant of $\underline{A}$ is

$$|\underline{A}|=a_{11}(a_{22}a_{33}-a_{23}^2)+a_{12}(a_{13}a_{23}-a_{12}a_{33}) +a_{13}(a_{12}a_{23}-a_{13}a_{22}) \ .$$

The inverse of $\underline{A}$ is

$$\underline{A}^{-1}=\underline{A}_c^T/|\underline{A}|$$

Let $\underline{A}=\underline{X}^T\underline{\omega}\underline{X}$ where

$$\underline{\omega}=\begin{bmatrix} 1 & & \underline{0} \\ & \ddots \; 1 & \\ \underline{0} & & \sigma^2/\sigma_{h_b}^2 \end{bmatrix} .$$

Thus by adding the prior information to the information in table 3, there results:

	Data Set 1	Data Set 2
$\sum_j X_{j2}^2\omega_j$	3.53161157	3.12700831
$\sum_j X_{j2}\omega_j Y_j$	127.0992273	122.5185789

and, for data set 1,

$$\underline{X}^T\underline{\omega}\underline{X}=\begin{bmatrix} 3.3250 & 1.6750 & 418{,}750 \\ & 3.53161157 & 418{,}750 \\ \text{symmetric} & & 83{,}340{,}625{,}000 \end{bmatrix}$$

or, for data set 2,

$$\underline{X}^T\underline{\omega}\underline{X}=\begin{bmatrix} 2.8500 & 1.6500 & 412{,}500 \\ & 3.12700831 & 412{,}500 \\ \text{symmetric} & & 8.3325\times10^{10} \end{bmatrix} .$$

Then, for data set 1,

$$\underline{A}_c=\begin{bmatrix} 1.18975153\times10^{11} & 3.575601563\times10^{10} & -777{,}456.0949 \\ & 1.017560156\times10^{11} & -690{,}937.5 \\ \text{symmetric} & & 8.93698347 \end{bmatrix}$$

and

$$|\underline{X}^T\underline{\omega}\underline{X}|=1.299239702\times10^{11} \ .$$

Therefore

$$[\underline{X}^T\underline{\omega}\underline{X}]^{-1}=\begin{bmatrix} 0.9157290438 & 0.2752072275 & -5.983931169\times10^{-6} \\ & 0.7831966299 & -5.318014058\times10^{-6} \\ \text{symmetric} & & 6.878625596\times10^{-11} \end{bmatrix} .$$

Similarly, for data set 2

$$\underline{A}_c=\begin{bmatrix} 9.040171743\times10^{10} & 3.267\times10^{10} & -609{,}265.9279 \\ & 6.732\times10^{10} & -495{,}000 \\ \text{symmetric} & & 6.189473684 \end{bmatrix}$$

and

$$|\underline{X}^T\underline{\omega}\underline{X}|=6.022819942\times10^{10} \ .$$

Therefore

$$[\underline{X}^T\underline{\omega}\underline{X}]^{-1}=\begin{bmatrix} 1.500986553 & 0.5424369368 & -1.011595787\times10^{-5} \\ & 1.117748839 & -8.218741466\times10^{-6} \\ \text{symmetric} & & 1.027670384\times10^{-10} \end{bmatrix} .$$

c. By adding the prior information to the information in table 3, we obtain for $\underline{X}^T\underline{\omega}\underline{Y}$:

	Data Set 1	Data Set 2
$\sum_j X_{j1}\omega_j Y_j$	192.18350	168.2030
$\sum_j X_{j2}\omega_j Y_j$	127.0992273	122.5185789
$\sum_j X_{j3}\omega_j Y_j$	26,879,687.5	26,583,550

By evaluating $(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{X}^T\underline{\omega}\underline{Y}$, estimates $\underline{\hat{b}}$ are obtained as:

	Data Set 1	Data Set 2
$\hat{b}_1$	50.12043881	50.01097198
$\hat{b}_2$	9.487418691	9.701194703
$\hat{b}_3$	0.00002302475114	0.00002342971729

Problem 3.3–1

The answers to parts a, b, and c are found in the section at the end of the problem where aids in debugging the computer code are given. The authors' computer code and output are:

```
      DIMENSION T(10),S(10),F(10),Z(2,10),C(2,2),G(2),D(2)
   10 FORMAT (I5,6F10.0,I5)
   20 FORMAT (1H1,35HNO. OF OBSERVATIONS (N) -------- = ,I7
     $/1H ,35HPUMPING RATE (Q) --------------- = ,G11.5
     $/1H ,35HDISTANCE FROM WELL CENTER (R) -- = ,G11.5
     $/1H ,35HINITIAL TRANSMISSIVITY (TO) ---- = ,G11.5
     $/1H ,35HINITIAL STORAGE COEFFICIENT (SO) = ,G11.5
     $/1H ,35HDAMPING PARAMETER (AP) --------- = ,G11.5
     $/1H ,35HCLOSURE CRITERION (ER) --------- = ,G11.5
     $/1H ,35HMAXIMUM NO. OF ITERATIONS (ITMX) = ,I7)
   30 FORMAT (8F10.0)
   32 FORMAT (10HOTIMES (T))
   34 FORMAT (23HOOBSERVED DRAWDOWNS (S))
   36 FORMAT (15HOITERATION NO. ,I4
     $/36H CURRENT ESTIMATES OF PARAMETERS (B))
   38 FORMAT (28HOSOLUTION FAILED TO CONVERGE)
   40 FORMAT ((1H ,10(G11.5,2X)))
   42 FORMAT (19HOSOLUTION CONVERGED)
   44 FORMAT (29HOFINAL COMPUTED DRAWDOWNS (F))
   46 FORMAT (16HORESIDUALS (F-S))
   48 FORMAT (18HOERROR VARIANCE = ,G11.5)
   50 FORMAT (12HOVAR(T)   = ,G11.5/12H COV(T,S) = ,G11.5
     $/12H VAR(S)   = ,G11.5)
   52 FORMAT (26HOSCALED SENSITIVITIES (Z):/5H TO T)
   54 FORMAT (5H TO S)
C         READ AND PRINT INPUT DATA
      READ(5,10) N,Q,R,TO,SO,AP,ER,ITMX
      WRITE(6,20) N,Q,R,TO,SO,AP,ER,ITMX
      READ(5,30) (T(I),I=1,N)
      WRITE(6,32)
      WRITE(6,40) (T(I),I=1,N)
      READ(5,30) (S(I),I=1,N)
      WRITE(6,34)
      WRITE(6,40) (S(I),I=1,N)
      FU=R*R/4.
      FW=Q/12.5664
      DMAX=ER+1.
      DO 140 KNT=1,ITMX
      UTMP=FU*SO/TO
      WTMP=FW/TO
      DO 80 I=1,N
C         COMPUTE NEW DRAWDOWN (F(I))
C           COMPUTE U AND W(U) FIRST
      U=UTMP/T(I)
      WU=W(U)
C           THEN F(I)
      F(I)=WTMP*WU
C         COMPUTE SCALED SENSITIVITIES (Z(I,J))
      TMP=EXP(-U)
      Z(1,I)=WTMP*(TMP-WU)
   80 Z(2,I)=-WTMP*TMP
C         CHECK FOR CONVERGENCE
      IF(DMAX.LT.ER) GO TO 150
C         ASSEMBLE COEFFICIENT MATRIX (C(I,J)) AND
C         GRADIENT VECTOR (G(J))
      DO 86 J=1,2
      DO 84 I=J,2
      C(I,J)=0.
   84 C(J,I)=0.
   86 G(J)=0.
      DO 110 K=1,N
      TMP=S(K)-F(K)
      DO 100 J=1,2
      DO 90 I=J,2
      C(I,J)=Z(I,K)*Z(J,K)+C(I,J)
```

```
   90 C(J,I)=C(I,J)
  100 G(J)=Z(J,K)*TMP+G(J)
  110 CONTINUE
C         INVERT COEFFICIENT MATRIX
      DET=C(1,1)*C(2,2)-C(1,2)*C(2,1)
      TMP=C(1,1)
      C(1,1)=C(2,2)/DET
      C(2,2)=TMP/DET
      C(1,2)=-C(1,2)/DET
      C(2,1)=C(1,2)
C         COMPUTE PARAMETER DISPLACEMENTS (D(I)), MAX. DISPLACEMENT
C         (DMAX), AND PARAMETERS (TO AND SO)
      DMAX=0.
      DO 130 J=1,2
      D(J)=C(1,J)*G(1)+C(2,J)*G(2)
      TMP=ABS(D(J))
      IF(TMP.GT.DMAX) DMAX=TMP
  130 CONTINUE
      TO=TO*(1.+AP*D(1))
      SO=SO*(1.+AP*D(2))
C         PRINT PARAMETERS
      WRITE(6,36) KNT
      WRITE(6,40) TO,SO
  140 CONTINUE
      WRITE(6,38)
      GO TO 160
  150 WRITE(6,42)
C         PRINT DRAWDOWNS, RESIDUALS (F(I)-S(I)), AND SCALED
C         SENSITIVITIES
  160 WRITE(6,44)
      WRITE(6,40) (F(I),I=1,N)
      DO 170 I=1,N
  170 F(I)=F(I)-S(I)
      WRITE(6,46)
      WRITE(6,40) (F(I),I=1,N)
      WRITE(6,52)
      WRITE(6,40) (Z(1,I),I=1,N)
      WRITE(6,54)
      WRITE(6,40) (Z(2,I),I=1,N)
C         COMPUTE AND PRINT ERROR VARIANCE (VAR) AND
C         COVARIANCE MATRIX FOR PARAMETERS
      VAR=0.
      DO 180 I=1,N
  180 VAR=VAR+F(I)*F(I)
      VAR=VAR/(N-2.)
      WRITE(6,48) VAR
      C(1,1)=TO*C(1,1)*TO*VAR
      C(1,2)=TO*C(1,2)*SO*VAR
      C(2,2)=SO*C(2,2)*SO*VAR
      WRITE(6,50) C(1,1),C(1,2),C(2,2)
      STOP
      END
      FUNCTION W(X)
C         COMPUTE THE WELL FUNCTION OF X
      W=0.
      IF(X.GT.10.) GO TO 20
      W=-0.577216-ALOG(X)+X
      TERM=X
      DO 10 J=2,36
      RJ=J
      TERM=-TERM*X/RJ
      TMP=TERM/RJ
      W=W+TMP
      IF(ABS(TMP).LT.1.E-7) GO TO 20
   10 CONTINUE
   20 RETURN
      END
```

```
1NO. OF OBSERVATIONS (N) -------- =       7
 PUMPING RATE (Q) --------------- =  1.1600
 DISTANCE FROM WELL CENTER (R) -- =  175.00
 INITIAL TRANSMISSIVITY (TO) ---- =  .10000
 INITIAL STORAGE COEFFICIENT (SO) =  .50000E-03
 DAMPING PARAMETER (AP) --------- =  1.0000
 CLOSURE CRITERION (ER) --------- =  .10000E-01
 MAXIMUM NO. OF ITERATIONS (ITMX) =      10
OTIMES (T)
  480.00      1020.0      1500.0      2040.0      2700.0      3720.0      4920.0
OOBSERVED DRAWDOWNS (S)
  1.7100      2.2300      2.5400      2.7700      3.0400      3.2500      3.5600
OITERATION NO.    1
 CURRENT ESTIMATES OF PARAMETERS (B)
  .11188      .54748E-03
OITERATION NO.    2
 CURRENT ESTIMATES OF PARAMETERS (B)
  .11347      .55219E-03
OITERATION NO.    3
 CURRENT ESTIMATES OF PARAMETERS (B)
  .11349      .55221E-03
OSOLUTION CONVERGED
OFINAL COMPUTED DRAWDOWNS (F)
  1.6715      2.2521      2.5564      2.8012      3.0256      3.2832      3.5086
ORESIDUALS (F-S)
 -.38538E-01  .22079E-01  .16407E-01  .31217E-01 -.14393E-01  .33213E-01 -.51356E-01
OSCALED SENSITIVITIES (Z):
 TO T
 -.91882     -1.4679     -1.7630     -2.0026     -2.2234     -2.4779     -2.7014
 TO S
 -.75264     -.78421     -.79343     -.79866     -.80223     -.80527     -.80724
OERROR VARIANCE =  .14328E-02
OVAR(T)    =  .95030E-05
 COV(T,S) = -.11369E-06
 VAR(S)   =  .14595E-08
```

Problem 3.3-2

b.

$$
\underline{D}=\begin{bmatrix}
2T_1 & -T_1 & & -T_1 & & & & & \\
-T_1 & 2(T_1+T_2) & -T_2 & & -(T_1+T_2) & & & & \\
 & -T_2 & 2T_2 & & & -T_2 & & & \\
-T_1 & & & 4T_1 & -2T_1 & & & & \\
 & -(T_1+T_2) & & -2T_1 & 4(T_1+T_2) & -2T_2 & & -(T_1+T_2 & \\
 & & -T_2 & & -2T_2 & 4T_2 & & & -T_2 \\
 & & & & & & 1 & & \\
 & & & & -(T_1+T_2) & & & 2(T_1+T_2) & -T_2 \\
 & & & & & -T_2 & & -T_2 & 2T_2
\end{bmatrix}
$$

$$
\underline{h}=\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix}
\qquad
\underline{q}=\begin{bmatrix}
\tfrac{1}{2}a^2W_1 \\
\tfrac{1}{2}a^2(W_1+W_2) \\
\tfrac{1}{2}a^2W_2 \\
a^2W_1+T_1h_{B1} \\
a^2(W_1+W_2) \\
a^2W_2+aq_{B1} \\
h_{B1} \\
\tfrac{1}{2}a^2(W_1+W_2)+T_1h_{B1} \\
\tfrac{1}{2}a^2W_2+aq_{B1}
\end{bmatrix}
$$

c.

$$\underline{J}_1=\begin{bmatrix} h_2+h_4-2h_1 \\ h_1+h_5-2h_2 \\ 0 \\ h_1+2h_5-4h_4+h_{B1} \\ h_2+2h_4+h_8-4h_5 \\ 0 \\ 0 \\ h_5-2h_8+h_{B1} \\ 0 \end{bmatrix}$$

$$\underline{J}_2=\begin{bmatrix} 0 \\ h_3+h_5-2h_2 \\ h_2+h_6-2h_3 \\ 0 \\ h_2+2h_6+h_8-4h_5 \\ h_3+2h_5+h_9-4h_6 \\ 0 \\ h_5+h_9-2h_8 \\ h_6+h_8-2h_9 \end{bmatrix}$$

$$\underline{J}_3=\begin{bmatrix} \tfrac{1}{2}a^2 \\ \tfrac{1}{2}a^2 \\ 0 \\ a^2 \\ a^2 \\ 0 \\ 0 \\ \tfrac{1}{2}a^2 \\ 0 \end{bmatrix} \qquad \underline{J}_4=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ 0 \\ 0 \\ a \end{bmatrix}$$

d.

1. Let $r=0$.
2. Compute $\underline{D}_r$ and $\underline{q}_r$.
3. $\underline{h}_r=\underline{D}_r^{-1}\,\underline{q}_r$.
4. Obtain $\underline{f}(\underline{\xi},\underline{b}_r)$ from $\underline{h}_r$ by deleting node 7.
5. $$\left.\begin{array}{l} \left(\dfrac{\partial \underline{h}}{\partial b_j}\right)_r=\underline{D}_r^{-1}\left[\left(\dfrac{\partial \underline{q}}{\partial b_j}\right)_r-\left(\dfrac{\partial \underline{D}}{\partial b_j}\right)_r\underline{h}_r\right] \\ \text{Obtain } \underline{X}_j^r \text{ from } \left(\dfrac{\partial \underline{h}}{\partial b_j}\right)_r \text{ by deleting node 7.} \end{array}\right\} j=1,2,\ldots,p$$
6. Compute $\underline{X}_r^T\underline{\omega}\underline{X}_r$ and $\underline{X}_r^T\underline{\omega}(\underline{Y}-\underline{f}(\underline{\xi},\underline{b}_r))$.
7. Define $\underline{C}_r=\{C^r_{ii}\}=\{(\underline{X}_r^T\underline{\omega}\underline{X}_r)_{ii}^{-1/2}\}$.
8. Compute $\underline{S}_r^T\underline{\omega}\underline{S}_r$ and $\underline{S}_r^T\underline{\omega}(\underline{Y}-\underline{f}(\underline{\xi},\underline{b}_r))$ where $\underline{S}_r=\underline{X}_r\underline{C}_r$.
9. $\underline{\delta}_{r+1}=(\underline{S}_r^T\underline{\omega}\underline{S}_r+\mu\underline{I})^{-1}\underline{S}_r^T\underline{\omega}(\underline{Y}-\underline{f}(\underline{\xi},\underline{b}_r))$
10. $\underline{d}_{r+1}=\underline{C}_r\underline{\delta}_{r+1}$.
11. $\underline{b}_{r+1}=\rho\underline{d}_{r+1}+\underline{b}_r$.
12. If $|d_i^{r+1}/c|>\epsilon$ (where $c=b_i^r$ for $b_i^r\neq 0$, and $c=1$ for $b_i^r=0$) for any $i=1,2,\ldots,p$, then increment r by one and return to 2. If not then:
13. $\underline{h}_{\text{final}}=\underline{D}_{r+1}^{-1}\underline{q}_{r+1}$.

Values of μ and ρ can be computed at each iteration by using algorithms defined by equations 3.3-28 through 3.3-30 if desired.

Problem 4.2-1

Data Set 1

Nodes in columns 2 through 11 have observations. Nodes in columns 1 and 12 form specified head boundaries. Spacing: Cell row 1, 1 ft; cell columns 1 and 11, 50 ft; cell columns 2 through 10, 100 ft (figure 1).

Data Set 2.

Nodes in columns 2 through 10 have observations. Nodes in columns 1 and 11 form specified head boundaries. Spacing: Cell row 1, 1 ft; cell columns 1 through 10, 100 ft (figure 2). The input data to the regression program for the two data sets are shown in figures 3 and 4.

If both T and W were estimated, the problem would be singular because the only unique parameter is W/T.

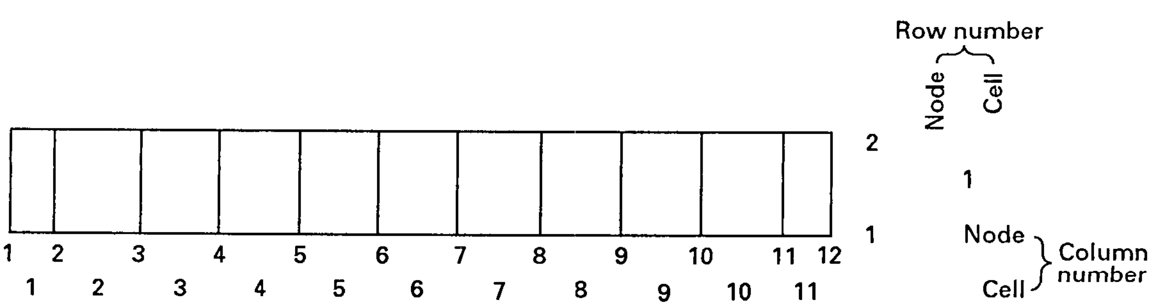


Figure 1

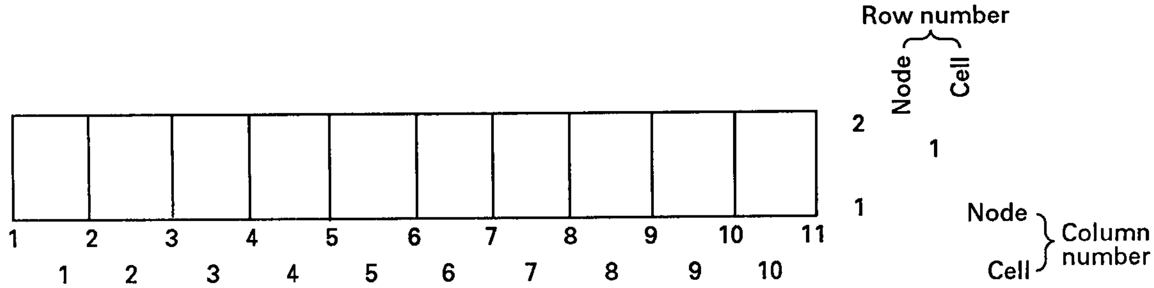


Figure 2

```
 LAKE OHPUPU, DATA SET 1.

    12     2     1    10     1     3     0     0     2     2     1     1     0
          50           0           0           0          .25
DX         1     0
     1    11     1     1           1     1
          50         100         100         100         100         100         100         100
         100         100          50
DY         1     0
     1     1     1     1           1
CX         1     0
     1    11     1     1           1
CY         1     0
     1    11     1     1           1
VL         1     0
     1    11     1     1           0
HR         1     0
     1    12     1     2           0
QR         1     0
     1    11     1     1           1
HC         1     0
     2    11     1     2          11
IZN        1     0
     1    11     1     1     1
     1     2     1          50          .5     48.33         1
     2     3     1         150          .5     45.76         1
     3     4     1         250          .5     42.08         1
     4     5     1         350          .5     38.34         1
     5     6     1         450          .5     35.30         1
     6     7     1         550          .5     31.00         1
     7     8     1         650          .5     25.85         1
     8     9     1         750          .5     21.76         1
     9    10     1         850          .5     16.11         1
    10    11     1         950          .5     12.48         1
     1     0     0     0     1
     1           0
     1           1           1           0       .001
IN         2     0
     1     1     1     2    -1
    12    12     1     2    -1
     1     2     2     2           0           0
     1     1          49
     1     2          49
     2     2     3     3         1.1         1.1
    12     1          11
    12     2          11
```

Figure 3

```
   LAKE OHPUPU, DATA SET 2.

    11    2    1    9    1    3    0    0    2    2    1    1    0
         50         0         0         0        .25
DX        1    0
     1   10    1    1       100
DY        1    0
     1    1    1    1         1
CX        1    0
     1   10    1    1         1
CY        1    0
     1   10    1    1         1
VL        1    0
     1   10    1    1         0
HR        1    0
     1   11    1    2         0
QR        1    0
     1   10    1    1         1
HC        1    0
     2   10    1    2        10
IZN       1    0
     1   10    1    1    1
     1    2    1       100        .5      47.13         1
     2    3    1       200        .5      44.14         1
     3    4    1       300        .5      39.89         1
     4    5    1       400        .5      36.36         1
     5    6    1       500        .5      32.48         1
     6    7    1       600        .5      29.70         1
     7    8    1       700        .5      24.33         1
     8    9    1       800        .5      19.10         1
     9   10    1       900        .5      14.96         1
     1    0    0    0    1
     1         0
     1         1         1         1        .001
IN        2    0
     1    1    1    2   -1
    11   11    1    2   -1
     1    2    2    2         0         0
     1    1        49
     1    2        49
     2    2    3    3       .95       .95
    11    1       9.5
    11    2       9.5
```

Figure 4

Problem 4.2-2

Because prior information is available, all parameters can be estimated. Data points are located in areas of relatively high sensitivity for all parameters. More data points in areas of highest sensitivity might improve results for parameters having low sensitivity. The input data to the regression program are shown in figure 1.

```
            CLASS PROBLEM
            TWO-DIMENSIONAL FLOW
            SEVERAL ZONES
   15   16    4   32    7   14    0    4    2   10    1    1    0
         2       .08        0         0         1
DX       1    0
    1   14    1    1         1    1
      1000      1000      1000      1000      1000      1000      1000       400
      1000      1000      1000      1000      1000      1000
DY       1    0
    1   15    1    1         1    1
      1000      1000      1000      1000      1000       400      1000      1000
      1000      1000      1000      1000      1000      1000      1000
CX       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
CY       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
VL       2    0
    8   14    6    6         1    0
    8    8    7   15         1    0
HR       2    0
    8   15    6    7       4.5    0
    8    9    7   16       4.5    0
QR       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
HC       1    0
    1   15    1   16         0    0
IZN      7    0
    6   14    1    8    2    0
    1   14    9   15    2    0
    1    6   12   15    1    0
    6    7    1    3    3    0
    8   14    1    4    3    0
    8   14    6    6    4    0
    8    8    7   15    4    0
    1    8    2      7000      1000     60.70         1
    2   14    2     12400      1000     75.64         1
    3   12    3     10400      2000     60.27         1
    4   10    4      8400      3000     29.67         1
    5    7    5      6000      4000      4.22         1
    6   11    5      9400      4000      4.37         1
    7   13    5     11400      4000      6.07         1
    8   14    5     13400      4000      5.81         1
    9   10    7      8400      5400      4.57         1
   10    8    8      7000      6400      5.21         1
   11   12    8     10400      6400    -44.89         1
   12   14    8     13400      6400      7.01         1
   13    7    9      6000      7400      6.95         1
   14    4   10      3000      8400     12.21         1
   15    9   10      7400      8400      4.04         1
```

Figure 1

```
   16   11   10      9400      8400    -89.36         1
   17    7   11      6000      9400      6.68         1
   18   13   11     11400      9400    -15.32         1
   19    3   12      2000     10400     16.88         1
   20    5   12      4000     10400     15.87         1
   21    9   12      7400     10400      4.48         1
   22   11   12      9400     10400    -18.34         1
   23   13   13     11400     11400     -2.47         1
   24   14   13     13400     11400      8.10         1
   25    3   14      2000     12400     54.12         1
   26    5   14      4000     12400     38.27         1
   27   10   14      8400     12400      .053         1
   28   12   14     10400     12400     -2.92         1
   29    7   15      6000     13400      8.30         1
   30   14   15     12400     13400      4.54         1
   31    2   15      1000     14400     85.82         1
   32   11   15      9400     14400      2.26         1
    1    8    8    0    9
    2   10   10    0   11
    3   12   12    0   13
    4   10   10   14    0
    8         0
    9    .00012
   10        84
   11         0
   12         0
   13   .000051
   14      .008
    1        70        70         0     .0004
    2       420       420         0    -.0002
    3        15        15         0    .00017
    4       420       420       .08         0
    1   16    7   16    1        45         0       .01
    7   16   15   16    2       420         0    .00056
   11   10   11   10    3    -97000      1940         1
   12    8   12    8    4    -51000      1020         1
IN       1    0
   15   15    5   16   -1    0
    1   10    5    6      1.04       .48
   15   16      10.4
   15   15
   15   14
   15   13
   15   12
   15   11
   15   10
   15    9
   15    8
   15    7       4.8
    2    2    6    7       .48       .54
   15    6       4.8
   15    5       5.4
```

Figure 1

Problem 5.4-1

a. Let $\underline{b}_0=\underline{0}$. Then

$$s^2=\frac{\underline{Y}^T\underline{\omega}\underline{Y}-\underline{\hat{b}}^T\underline{X}^T\underline{\omega}\underline{Y}}{n-p}$$

$$\underline{Y}=\begin{bmatrix}48.33\\45.76\\42.08\\38.34\\35.30\\31.00\\25.85\\21.76\\16.11\\12.48\\11\end{bmatrix}\text{ for data set 1, or }\begin{bmatrix}47.13\\44.14\\39.89\\36.36\\32.48\\29.70\\24.33\\19.10\\14.96\\9.5\end{bmatrix}\text{ for data set 2}$$

$$\underline{\omega}=\begin{bmatrix}1&&&&\\&1&&&\\&&\cdots&&\\&&&1&\\&&&&\frac{0.25}{(1.1)^2}\end{bmatrix}\text{ or }\begin{bmatrix}1&&&&\\&1&&&\\&&\cdots&&\\&&&1&\\&&&&\frac{0.25}{(0.95)^2}\end{bmatrix}$$

$$\underline{\hat{b}}=\begin{bmatrix}50.12043881\\9.487418691\\0.00002302475114\end{bmatrix}\text{ or }\begin{bmatrix}50.01097198\\9.701194703\\0.00002342971729\end{bmatrix}$$

$$\underline{X}^T\underline{\omega}\underline{Y}=\begin{bmatrix}192.18350\\127.0992273\\26{,}879{,}687.5\end{bmatrix}\text{ or }\begin{bmatrix}168.2030\\122.5185789\\26{,}583{,}550\end{bmatrix}$$

$\underline{Y}^T\underline{\omega}\underline{Y}=$ 11,459.5411 or 10,225.4391

$\underline{\hat{b}}^T\underline{X}^T\underline{\omega}\underline{Y}=$11,457.06305 or 10,223.41717

$s^2=$ 0.30975625 or 0.2888471429

$s/\Delta Y_s=$ 0.5565574993/(48.33-12.48)=0.0155 or
0.5374450138/(47.13-14.96)=0.0167

The fit is fairly good.

b.

$$(\underline{X}^T\underline{\omega}\underline{X})^{-1}s^2=$$

$$0.30975625\begin{bmatrix}0.9157290438 & 0.2752072275 & -5.983931169\times10^{-6}\\ & 0.7831966299 & -5.318014058\times10^{-6}\\ \text{symmetric} & & 6.878625596\times10^{-11}\end{bmatrix}$$

$$=\begin{bmatrix}0.2836527946 & 0.08524715876 & -1.853560079\times10^{-6}\\ & 0.2426000511 & -1.647288092\times10^{-6}\\ \text{symmetric} & & 2.130697270\times10^{-11}\end{bmatrix}$$

or

$$0.2888471429\begin{bmatrix}1.500986553 & 0.5424369368 & -1.011595787\times10^{-5}\\ & 1.117748839 & -8.218741466\times10^{-6}\\ \text{symmetric} & & 1.027670384\times10^{-10}\end{bmatrix}$$

$$=\begin{bmatrix}0.4335556774 & 0.1566813594 & -2.921965528\times10^{-6}\\ & 0.3228585586 & -2.373959991\times10^{-6}\\ \text{symmetric} & & 2.968396543\times10^{-11}\end{bmatrix}.$$

Because the standard errors (the square roots of the diagonal elements) are small, the parameters are well determined.

c.

$$\underline{r} = \begin{bmatrix} 1 & 0.3249682531 & -0.7539668635 \\ & 1 & -0.7245414573 \\ \text{symmetric} & & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0.4187827075 & -0.8145011290 \\ & 1 & -0.7668426954 \\ \text{symmetric} & & 1 \end{bmatrix}.$$

The problem is fairly well conditioned.

Problem 5.5-1

The plots of $\underline{d}$ and $\underline{g}$ (figures 1,2) suggest that correlation manifests itself in two ways: The variability of $\underline{g}$ from set to set is smaller than the variability of $\underline{d}$ from set to set, and the plots of $\underline{g}$ do not appear to be linear on normal probability paper. The plot of $\underline{\hat{e}}$ (figure 3) does not appear to differ very much from either the plots of $\underline{d}$ or $\underline{g}$, although the plot of $\underline{\hat{e}}$ does show the same nonlinear trend as displayed by the plots of $\underline{g}$. Therefore, one may say that the plot of $\underline{\hat{e}}$ does not appear to differ significantly from the plot of a $N(\underline{0},(\underline{I}-\underline{R})s^2)$ random variable. Furthermore, this distribution of residuals suggests that the Theis model is adequate to describe the observed-drawdown data set. Input data for the residuals analysis program are shown in figure 4.

Problem 5.5-2

a. $s^2 = 0.98677$

$R_y = 0.99964$

$s/\Delta Y_s = 0.99336297/175.18 = 0.00567$

b. The problem is not well conditioned. Entries (1,8), (1,9), (8,9), and (12,13) (and their symmetric counterparts) of the scaled least squares matrix have absolute values greater than 0.9, although these large values yielded large (absolute value >0.9) parameter correlations only for entries (1,8) and (12,13). The correlation between parameters 12 and 13 is exceptionally large so that these parameters are behaving as one. Because parameters 12 and 13 are the transmissivity T_3 and recharge W_3 in zone 3, respectively, the physical interpretation is that the ratio W_3/T_3 is much more unique than either W_3 or T_3. The only poorly determined parameter is the flow across boundary zone 2, q_{B2}. Because of the large value of transmissivity in the aquifer zone adjacent to this boundary flow zone, the gradient near the boundary is low. Hence, the flow across the boundary is estimated poorly. Well discharges Q_1 and Q_2 and transmissivity T_2 are estimated quite well. The drawdown cones provide large gradients that serve to determine Q_1, Q_2, and T_2 precisely.

c. Effects of correlation within $\underline{g}$ are not large. They may be exhibited as a slight steepening of the curves for $\underline{g}$ (figure 2) compared to those for $\underline{d}$ (figure 1) on the normal probability plots. The plot of $\underline{\hat{u}}$ (figure 3) is very similar to those of $\underline{g}$, which suggests that the distribution of $\underline{\hat{u}}$ does not differ significantly from a $N(\underline{0},(\underline{I}-\underline{R})s^2)$ distribution. Note that $\underline{\hat{e}}$ could not have been used instead of $\underline{\hat{u}}$ to make the comparison because the weight matrix $\underline{\omega}$ is not equal to $\underline{I}$. The plot of $\hat{u}_j$ versus $\bar{f}_j$ (figure 4) shows no pattern. However, the plot of $\hat{e}_j$ versus Cartesian coordinate (figure 5) shows a group of negative residuals in the upper center of the area. This sign pattern was inherited from the sign pattern of the original errors $\underline{\epsilon}$ that were generated (recall that this exercise is based on a hypothetical problem), and the original errors are random $N(0,1)$ deviates. Hence, the sign pattern occurred entirely by chance. The lesson is that apparently nonrandom patterns can, and often do, develop by chance, and the analyst must learn to distinguish true problem areas from apparent ones.

The input data for the residuals analysis program are shown in figure 6.

Ordered residual distribution: (The calculated values are prior information residuals $\hat{u}_j = \hat{e}_j \omega_j^{1/2}$, where $\omega_j^{1/2} = s/(\mathrm{Var}(\epsilon_{pj}))^{1/2}$.)

No.	$\hat{u}_j$	No.	$\hat{u}_j$
1	-2.3900	21	-0.013826
2	-1.4247	22	$(-97000+97000)5.1204\times10^{-4}= \approx 0$
3	-1.3248	23	0.034151
4	-1.2986	24	0.036783
5	-1.1685	25	$(-50961+51000)9.7389\times10^{-4}=0.037982$
6	-0.95063	26	(0.080716-0.08)124.17=0.088906
7	(0.00031149-0.0004)8278.0=-0.73269	27	(5.4730-5.4)1.8396=0.13429
8	-0.72189	28	0.22868
9	-0.70275	29	0.24557
10	(0.00013516-0.00017)19478=-0.67861	30	0.33619
11	-0.50751	31	0.40574
12	-0.43737	32	0.41665
13	-0.43737	33	0.64095
14	-0.35588	34	0.65517
15	-0.33702	35	(5.1211-4.8)2.0695=0.66452
16	-0.26802	36	(487.89-420)0.011826=0.80287
17	(10.198-10.4)0.95516=-0.19294	37	0.90021
18	-0.18186	38	0.98513
19	-0.087697	39	1.1503
20	-0.080833	40	1.3703
		41	1.6158

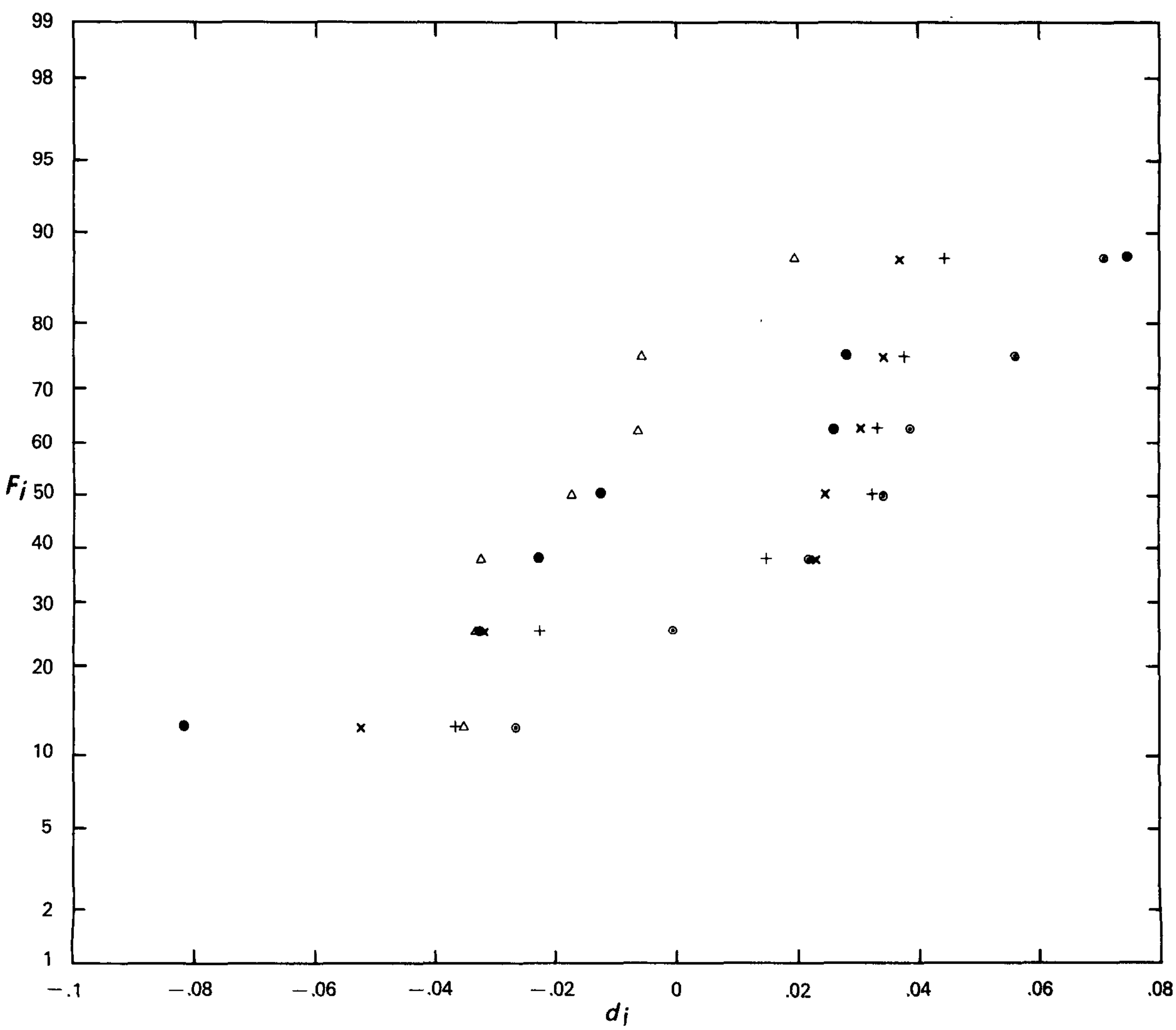


Figure 1

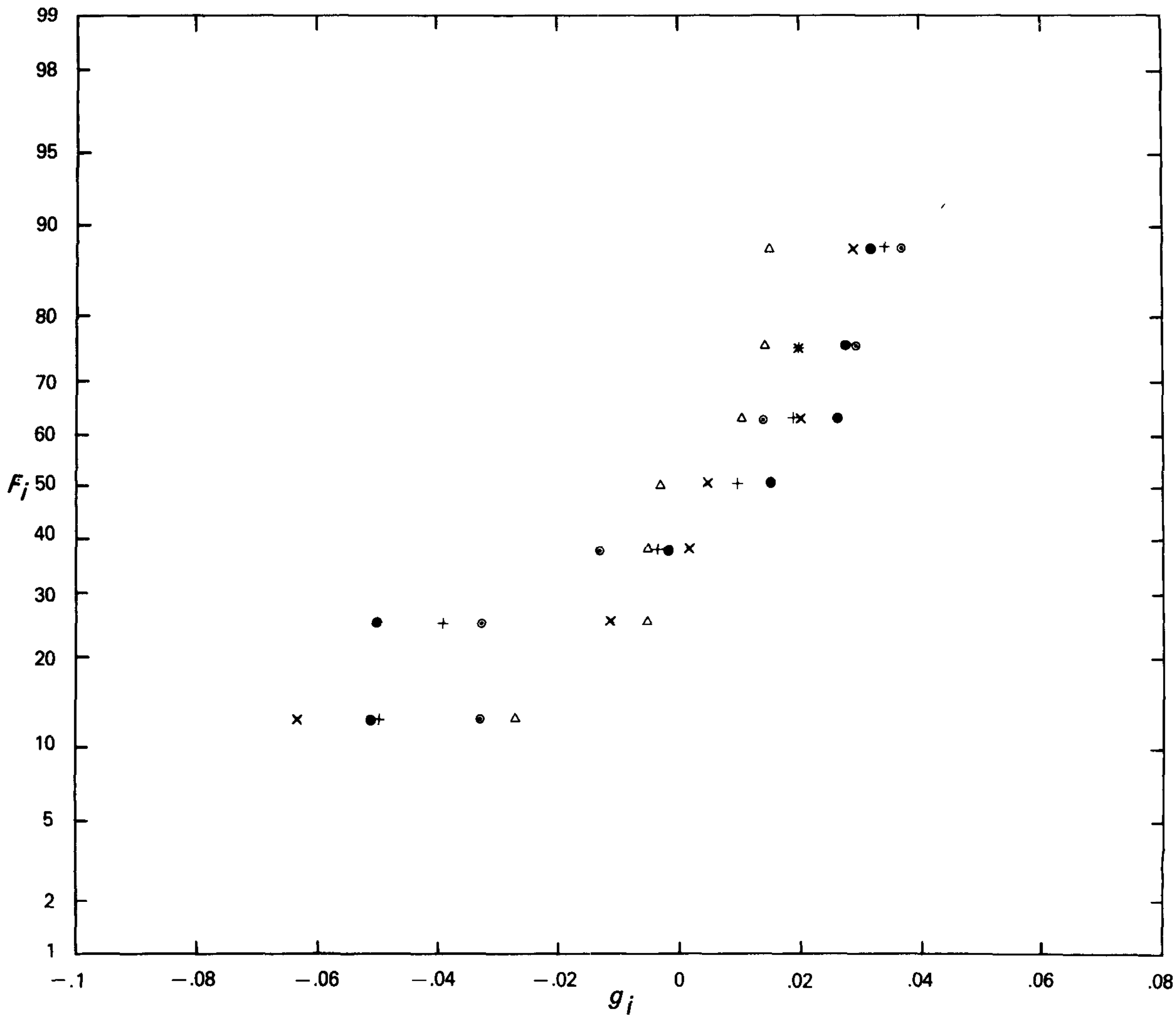


Figure 2

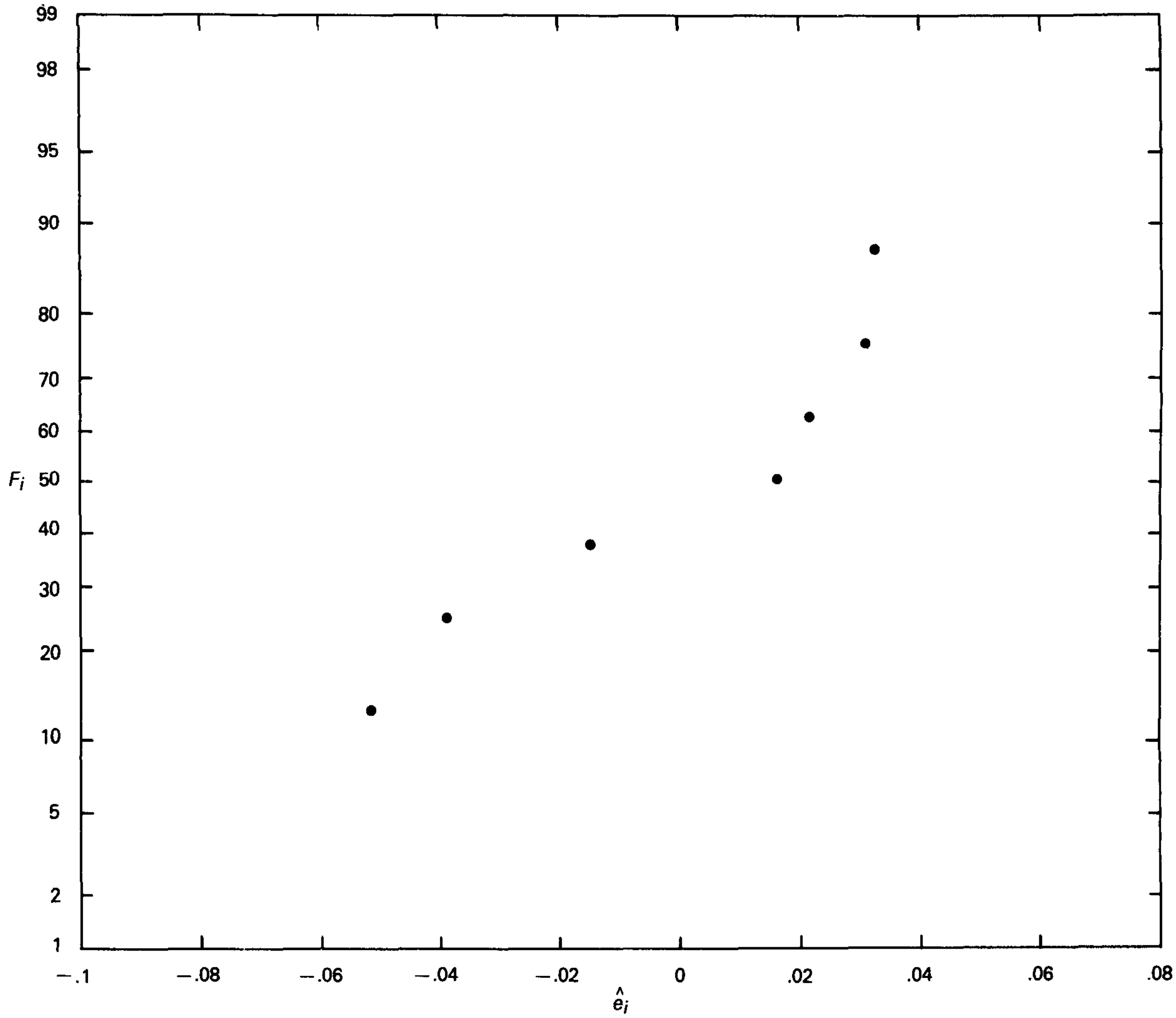


Figure 3

```
  2    7    0    5    1  .0014328
.95030E-5-.11369E-6
.14595E-8
          1          1          1          1          1          1          1
 -8.09604  -1362.96
 -12.9342  -1420.13
 -15.5344  -1436.83
 -17.6456  -1446.30
 -19.5912  -1452.76
 -21.8336  -1458.27
 -23.8030  -1461.84
```

Figure 4

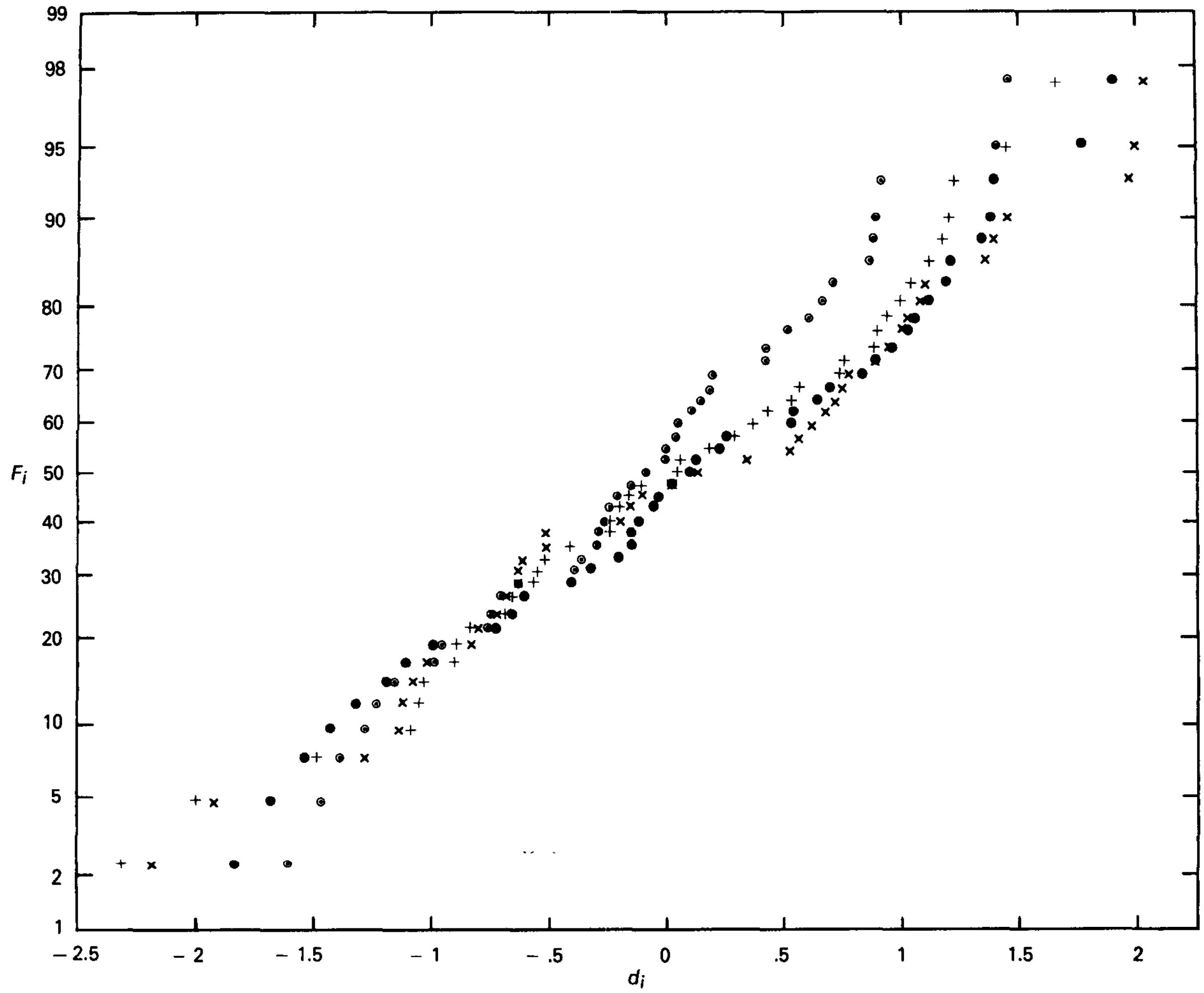


Figure 1

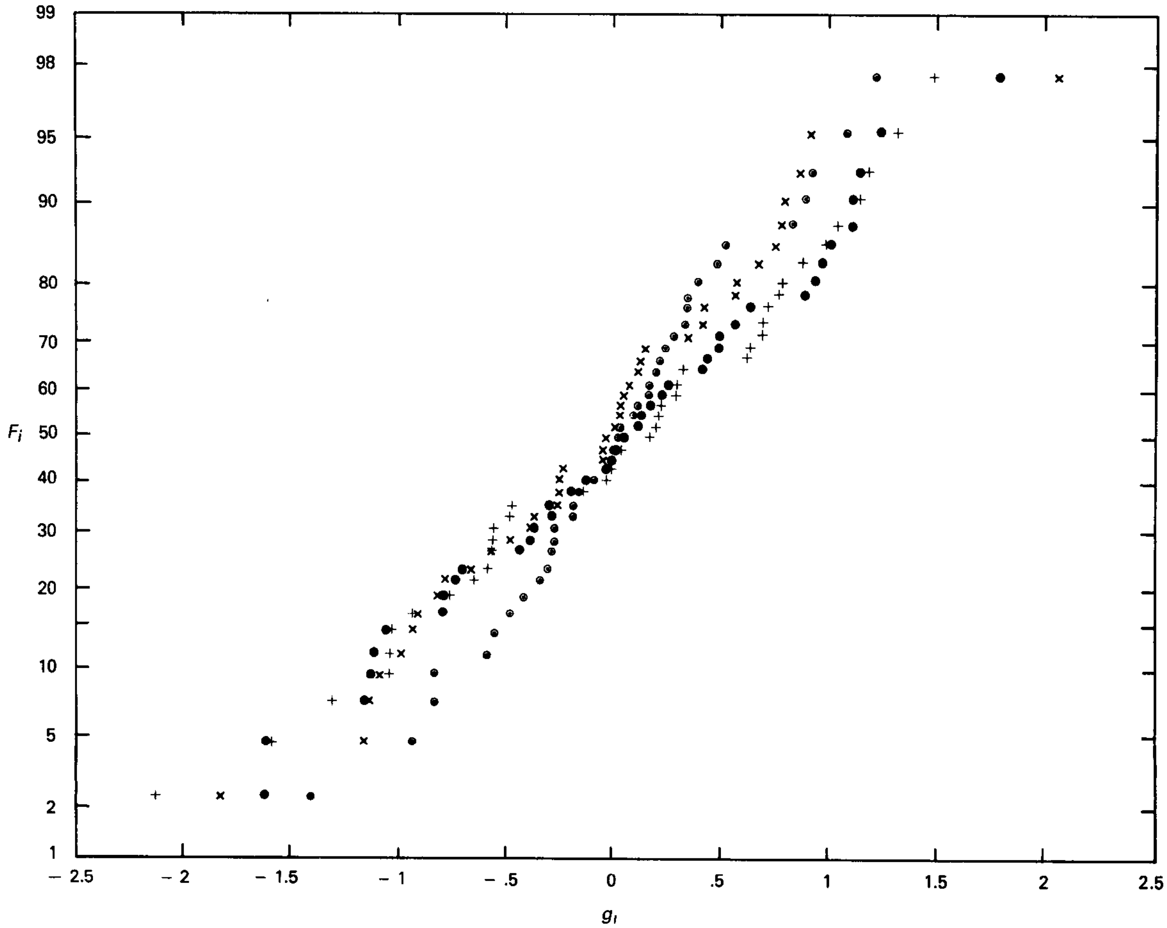


Figure 2

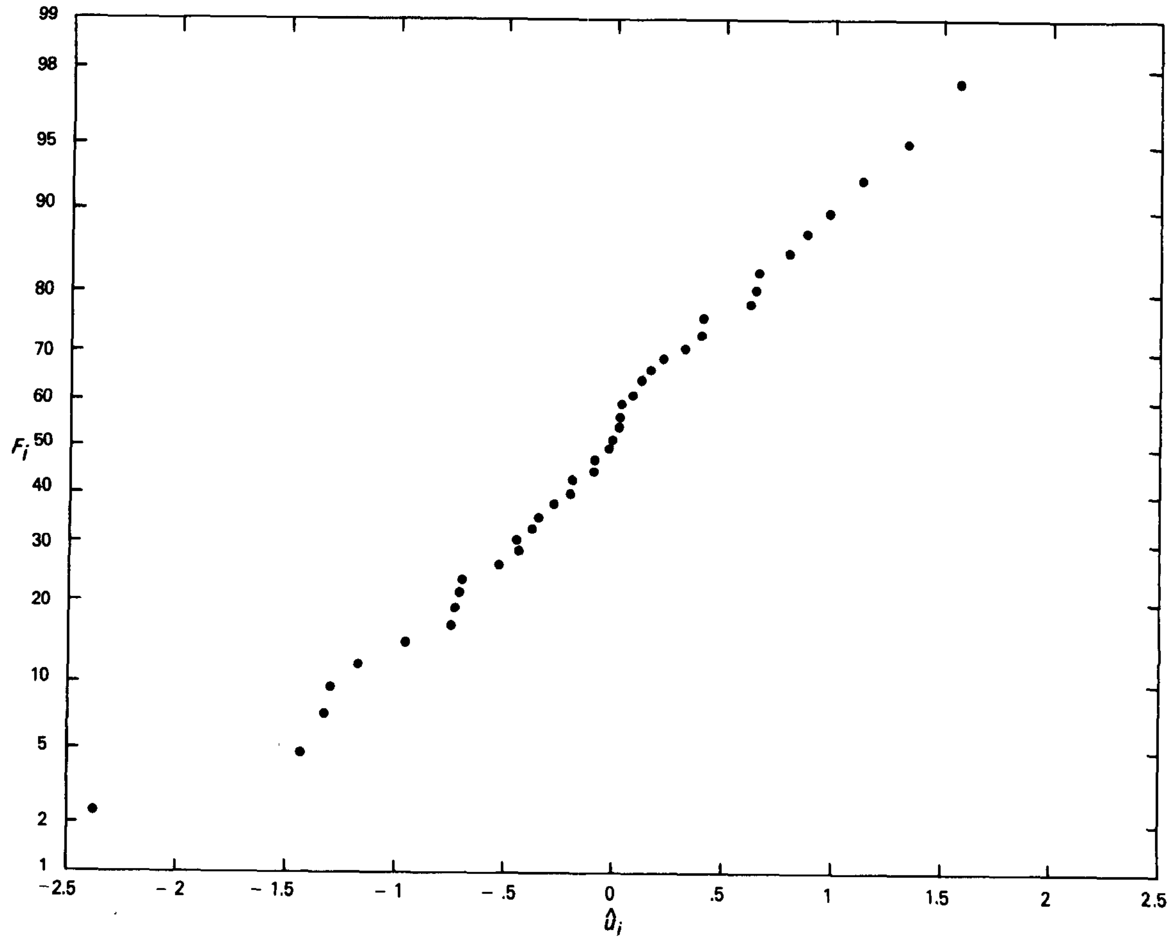


Figure 3

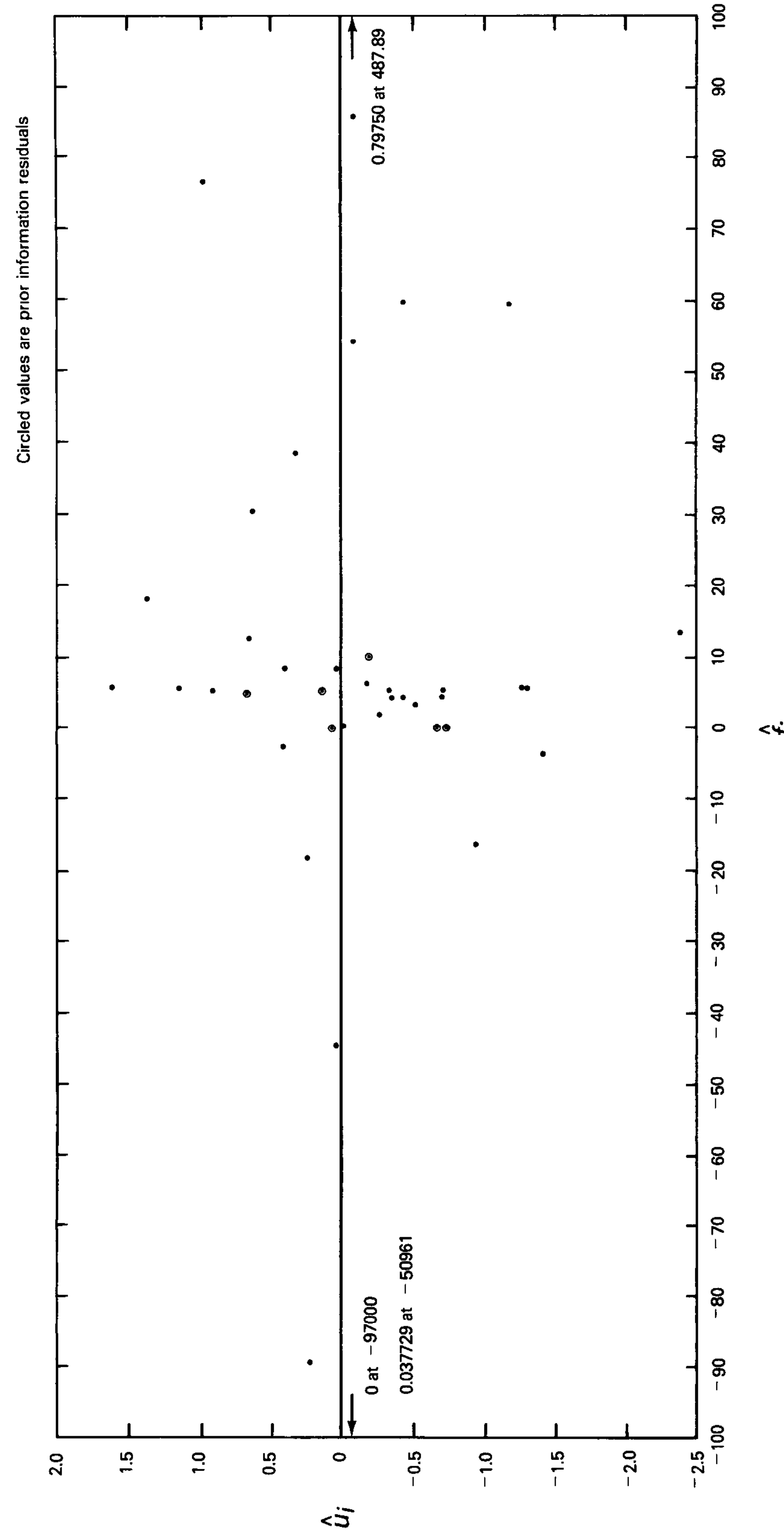

Circled values are prior information residuals
0.79750 at 487.89
0 at −97000
0.037729 at −50961
$\hat{u}_i$
$\hat{f}_i$
2.0
1.5
1.0
0.5
0
−0.5
−1.0
−1.5
−2.0
−2.5
−100
−90
−80
−70
−60
−50
−40
−30
−20
−10
0
10
20
30
40
50
60
70
80
90
100

Figure 4

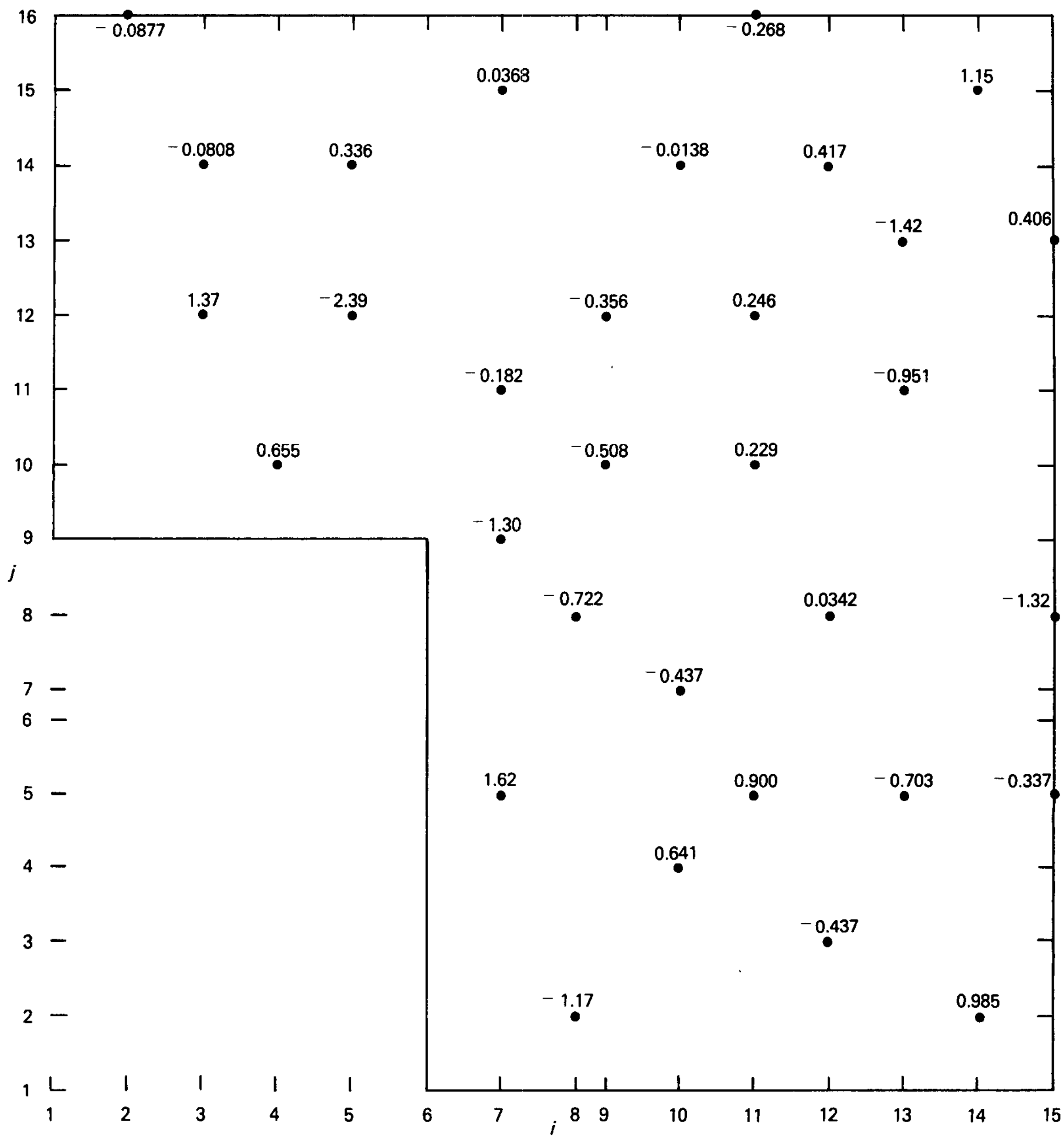


Figure 5

```
14    32     9     4   437     .98677
       1
 3     4     5     6     7     9    10    13    14
   1940       1020       1.04        .48        .54    .00012        84    .000051
   .008
```

Figure 6

Problem 5.6-1

a. Compute

$\underline{H}(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{H}^T$

where $\underline{H}$=[0 0 1]. Then,

$\underline{H}(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{H}^T=\widehat{\text{Var}}(\hat{b}_3)/s^2$.

By using equation 5.6-11,

$$W=\frac{(\tilde{\beta}_3-\hat{b}_3)(\text{Var}(\hat{b}_3)/s^2)^{-1}(\tilde{\beta}_3-\hat{b}_3)}{s^2}$$

$$=\frac{(\tilde{\beta}_3-\hat{b}_3)^2}{\widehat{\text{Var}}(\hat{b}_3)}$$

$$H_0{:}\beta_3=\frac{0.0003}{10}=3\times10^{-5}$$

$H_1{:}\beta_3 \neq 3\times10^{-5}$

$$w=\frac{(3\times10^{-5}-2.3025\times10^{-5})^2}{2.1307\times10^{-11}}$$

= 2.273 for data set 1.

$F_{0.05}(1,8)=5.318$. ∵ H_0 accepted

Values of the ratio of recharge to transmissivity, $\beta_3=W/T$, computed by the two methods are not significantly different.

$$w=\frac{(3\times10^{-5}-2.3430\times10^{-5})^2}{2.9684\times10^{-11}}$$

= 1.454 for data set 2

$F_{0.05}(1,7)=5.591$. ∵ H_0 accepted.

The Maxey-Eakin estimate of W/T could be used as prior information in the regression model, but an estimate of Var(W/T) would also be needed.

b. This test is the same as the one in a except that

$H_0{:}\beta_3=0$ $H_1{:}\beta_3\neq0$

$$w=\frac{(0-2.3025\times10^{-5})^2}{2.1307\times10^{-11}}$$

= 24.88 for data set 1

∵ H_0 rejected.

$$w=\frac{(0-2.3430\times10^{-5})^2}{2.9684\times10^{-11}}$$

= 18.49 for data set 2

∵ H_0 rejected.

W/T is significantly different from zero. Thus, recharge is a significant variable in the regression equation.

c. $\tilde{\beta}_3=\hat{b}_3\pm\sqrt{qF_\alpha(q,n-p)}s_{b3}$

$=2.3025\times10^{-5}\pm\sqrt{5.318}\times4.6159\times10^{-6}$

$=2.3025\times10^{-5}$
$\pm1.0645\times10^{-5}$ for data set 1 .

$\tilde{\beta}_3=2.3430\times10^{-5}\pm\sqrt{5.591}\times5.4483\times10^{-6}$

$=2.3430\times10^{-5}$
$\pm1.2883\times10^{-5}$ for data set 2.

Problem 5.6-2

For T extreme:

$$T=\hat{T}\pm\frac{\sqrt{2F_{0.05}(2,5)}}{s_{bT}}\widehat{\text{Var}}(T)$$

$$=0.11349\pm\frac{\sqrt{2\times5.7861}}{0.0030827}(0.95030\times10^{-5})$$

$=0.11349\pm0.010487$

$$S=\hat{S}\pm\frac{\sqrt{2F_{0.05}(2,5)}}{s_{bT}}\widehat{\text{Cov}}(T,S)$$

$$=0.00055221\pm\frac{\sqrt{2\times5.7861}}{0.0030827}(-0.11369\times10^{-6})$$

$=0.00055221\mp0.00012546.$

For S extreme:

$$T=\hat{T}\pm\frac{\sqrt{2F_{0.05}(2,5)}}{s_{bS}}\widehat{\mathrm{Cov}}(T,S)$$

$$=0.11349\pm\frac{\sqrt{2\times 5.7861}}{3.8203\times 10^{-5}}(-0.11369\times 10^{-6})$$

$$=0.11349\mp 0.010124$$

$$S=\hat{S}\pm\frac{\sqrt{2F_{0.05}(2,5)}}{s_{bS}}\widehat{\mathrm{Var}}(S)$$

$$=0.00055221\pm\frac{\sqrt{2\times 5.7861}}{3.8203\times 10^{-5}}(0.14595\times 10^{-8})$$

$$=0.00055221\pm 0.00012996\,.$$

Problem 5.6-3

$$\underline{\tilde{b}}=\underline{\hat{b}}\pm\frac{\sqrt{qF_\alpha(q,n-p)}}{s_{bi}}\underline{V}_{bi}$$

where $\underline{V}_b=(\underline{X}^T\underline{\omega}\underline{X})^{-1}s^2$. Let $(\underline{X}^T\underline{\omega}\underline{X})^{-1}s^2=\underline{A}=\{A_{ij}\}$. Then the above equation can be written in algebraic form for calculations as

$$\tilde{b}_j=\hat{b}_j\pm\sqrt{qF_\alpha(q,n-p)}\ \frac{A_{ij}}{\sqrt{A_{ii}}}\ .$$

Calculations:

$$\sqrt{qF_\alpha(q,n-p)}=\sqrt{2F_{0.05}(2,27)}=\sqrt{2\times 3.359}=2.592$$

For $T_3=\beta_{12}$ extreme:

$\tilde{b}_1=80.978\pm 9.2811$
$\tilde{b}_2=935.15\pm 208.42$
$\tilde{b}_3=-97000\mp 85.314$
$\tilde{b}_4=-50961\mp 29.377$
$\tilde{b}_5=10.198\pm 0.11520$
$\tilde{b}_6=5.1211\pm 1.3758\times 10^{-2}$
$\tilde{b}_7=5.4730\pm 1.2758\times 10^{-2}$
$\tilde{b}_8=65.754\pm 7.2341$
$\tilde{b}_9=3.1149\times 10^{-4}\pm 3.0988\times 10^{-5}$
$\tilde{b}_{10}=487.89\pm 2.5660$
$\tilde{b}_{11}=-1.3995\times 10^{-4}\mp 4.9215\times 10^{-5}$
$\tilde{b}_{12}=13.288\pm 9.3998$
$\tilde{b}_{13}=1.3516\times 10^{-4}\pm 9.3069\times 10^{-5}$
$\tilde{b}_{14}=8.0716\times 10^{-2}\pm 3.5370\times 10^{-5}$

For $q_{B1}=\beta_1$ extreme:

$\tilde{b}_1=80.978\pm 43.911$
$\tilde{b}_2=935.15\pm 489.14$
$\tilde{b}_3=-97000\mp 222.36$
$\tilde{b}_4=-50961\mp 73.632$
$\tilde{b}_5=10.198\pm 0.27770$
$\tilde{b}_6=5.1211\pm 3.2747\times 10^{-2}$
$\tilde{b}_7=5.4730\mp 3.2200\times 10^{-3}$
$\tilde{b}_8=65.754\pm 25.550$
$\tilde{b}_9=3.1149\times 10^{-4}\pm 5.4093\times 10^{-5}$
$\tilde{b}_{10}=487.89\pm 6.2288$
$\tilde{b}_{11}=-1.3995\times 10^{-4}\mp 1.1701\times 10^{-4}$
$\tilde{b}_{12}=13.288\pm 1.9832$
$\tilde{b}_{13}=1.3516\times 10^{-4}\pm 2.0377\times 10^{-5}$
$\tilde{b}_{14}=8.0716\times 10^{-2}\pm 1.5001\times 10^{-4}\,.$

Problem 5.7-1

a. Let $\underline{X}=\{X_{ij}\}$, $(\underline{X}^T\underline{\omega}\underline{X})^{-1}=\{A_{ij}\}$. Then

$$\underline{X}(\underline{X}^T\underline{\omega}\underline{X})^{-1}=\{\sum_{k=1}^{p}X_{ik}A_{k\ell}\}=\{C_{i\ell}\}$$

$$\underline{X}(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{X}^T=\{\sum_{\ell=1}^{p}C_{i\ell}X_{j\ell}\}=$$

$$\{\sum_{\ell=1}^{p}\sum_{k=1}^{p}X_{ik}A_{k\ell}X_{j\ell}\}=\{\sum_{\ell=1}^{p}X_{j\ell}\sum_{k=1}^{p}X_{ik}A_{k\ell}\}\,.$$

If $i=j$, then the entry is

$$\sum_{\ell=1}^{p}\sum_{k=1}^{p}X_{ik}A_{k\ell}X_{i\ell}=\sum_{\ell=1}^{p}X_{i\ell}\sum_{k=1}^{p}X_{ik}A_{k\ell}\,.$$

Compute for $i=j=1$ for data set 1.

$$\underline{X}_1=[0.95\ 0.05\ 23{,}750]\,.$$

$$(\underline{X}^T\underline{\omega}\underline{X})^{-1}=\begin{bmatrix}0.9157290438 & 0.2752072275 & -5.983931169\times10^{-6}\\ & 0.7831966299 & -5.318014058\times10^{-6}\\ \text{symmetric} & & 6.878625596\times10^{-11}\end{bmatrix}.$$

$$\begin{aligned}\underline{X}_1(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{X}_1^T&=0.95(0.95\times0.9157290438\\&+0.05\times0.2752072275\\&+23{,}750\times(-5.983931169\times10^{-6}))\\&+0.05(0.95\times0.2752072275\\&+0.05\times0.7831966299\\&+23{,}750\times(-5.318014058\times10^{-6}))\\&+23{,}750\times(0.95\times(-5.983931169\times10^{-6})\\&+0.05\times(-5.318014058\times10^{-6})\\&+23{,}750\times6.878625596\times10^{-11}))\\&=0.6106927103\,.\end{aligned}$$

$$\begin{aligned}\underline{X}(\underline{X}^T\underline{\omega}\underline{X})^{-1}\underline{X}^T&=\underline{X}\underline{C}\underline{C}^{-1}(\underline{X}^T\underline{\omega}\underline{X})^{-1}(\underline{C}\underline{C}^{-1})^T\underline{X}^T\\&=\underline{X}\underline{C}(\underline{C}^T\underline{X}^T\underline{\omega}\underline{X}\underline{C})^{-1}\underline{C}^T\underline{X}^T\\&=(\underline{X}\underline{C})((\underline{X}\underline{C})^T\underline{\omega}(\underline{X}\underline{C}))^{-1}(\underline{X}\underline{C})^T\\&=\underline{S}(\underline{S}^T\underline{\omega}\underline{S})^{-1}\underline{S}^T\,.\end{aligned}$$

b. $\widehat{\mathrm{Var}}(\hat{f}_j)=\underline{X}_j\widehat{\mathrm{Var}}(\hat{\underline{b}})\underline{X}_j^T$

Let $j=1$, data set 1.

$$\begin{aligned}\widehat{\mathrm{Var}}(\hat{f}_1)&=\underline{X}_1(\underline{X}^T\underline{\omega}\underline{X})^{-1}s^2\underline{X}_1^T\\&=0.6106927103\times0.30975625\\&=0.1891658838\,.\end{aligned}$$

c. $f_{\beta j}=\hat{f}_j\pm\sqrt{pF_\alpha(p,n-p)}s_{yj}$

$F_{0.05}(3,8)=4.066$

$$\begin{aligned}\hat{f}_1&=0.95\hat{b}_1+0.05\hat{b}_2+23{,}750\hat{b}_3\\&=0.95(50.1204)+0.05(9.48742)\\&+23{,}750(2.30248\times10^{-5})\\&=48.636\end{aligned}$$

$$\begin{aligned}f_{\beta1}&=48.636\pm\sqrt{3\times4.066}\times0.434932\\&=48.636\pm1.519\end{aligned}$$

for $j=1$, data set 1.

Problem 6.2-1

Sets of parameters for the modified Beale's measure:

1. (0.12398,0.00042675)
2. (0.10300,0.00067767)
3. (0.10337,0.00068217)
4. (0.12361,0.00042225)

Only 2 and 4 need be used because the other two are nearly the same. By using 2, drawdown, s, is: 1.5947, 2.2223, 2.5541, 2.8218, 3.0676, 3.3503, 3.5979. By using 4, s is: 1.7821, 2.3239, 2.6058, 2.8320, 3.0390, 3.2763, 3.4839.

The resulting value of the modified Beale's measure is

$$\hat{N}_b=0.027702$$

Because $F_{0.05}(2,5)=5.7861$, $0.09/F=0.0156$ and $1/F=0.173$, so that the model is almost roughly linear.

The input data for the modified Beale's measure program are shown in figure 1.

```
   2       2       7       0       2   .0014328
 .11349 .00055221
 1.6715    2.2521    2.5564    2.8012    3.0256    3.2832    3.5086
      1         1         1         1         1         1         1
-8.09604  -1362.96
-12.9342  -1420.13
-15.5344  -1436.83
-17.6456  -1446.30
-19.5912  -1452.76
-21.8336  -1458.27
-23.8030  -1461.84
 .10300 .00067767
 1.5947    2.2223    2.5541    2.8218    3.0676    3.3503    3.5979
 .12361 .00042225
 1.7821    2.3239    2.6058    2.8320    3.0390    3.2763    3.4839
```

Figure 1

Problem 6.2-2

From the computer output, the modified Beale's measure is 0.29808. Based on $F_{0.05}(2,27) = 3.359$, the modified Beale's measure indicates that the model is at the point of being highly nonlinear. Hence, linear theory can be applied to use the W statistic based on $q=2$ only as a very rough approximation.

The input data to the regression code, modified to compute the modified Beale's measure, are shown in figure 1.

```
            CLASS PROBLEM
            TWO-DIMENSIONAL FLOW
            SEVERAL ZONES
   15   16    4   32    7   14    0    4    2   10    0    0    0
         2      .08         0         0         1
DX       1    0
    1   14    1    1         1    1
      1000      1000      1000      1000      1000      1000      1000       400
      1000      1000      1000      1000      1000      1000
DY       1    0
    1   15    1    1         1    1
      1000      1000      1000      1000      1000       400      1000      1000
      1000      1000      1000      1000      1000      1000      1000
CX       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
CY       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
VL       2    0
    8   14    6    6         1    0
    8    8    7   15         1    0
HR       2    0
    8   15    6    7       4.5    0
    8    9    7   16       4.5    0
QR       2    0
    6   14    1    8         1    0
    1   14    9   15         1    0
HC       1    0
    1   15    1   16         0    0
IZN      7    0
    6   14    1    8    2    0
    1   14    9   15    2    0
    1    6   12   15    1    0
    6    7    1    3    3    0
    8   14    1    4    3    0
    8   14    6    6    4    0
    8    8    7   15    4    0
    1    8    2     7000      1000     60.70          1
    2   14    2    12400      1000     75.64          1
    3   12    3    10400      2000     60.27          1
    4   10    4     8400      3000     29.67          1
    5    7    5     6000      4000      4.22          1
    6   11    5     9400      4000      4.37          1
    7   13    5    11400      4000      6.07          1
    8   14    5    13400      4000      5.81          1
    9   10    7     8400      5400      4.57          1
   10    8    8     7000      6400      5.21          1
   11   12    8    10400      6400    -44.89          1
   12   14    8    13400      6400      7.01          1
   13    7    9     6000      7400      6.95          1
   14    4   10     3000      8400     12.21          1
   15    9   10     7400      8400      4.04          1
```

Figure 1

```
   16   11   10      9400      8400    -89.36         1
   17    7   11      6000      9400      6.68         1
   18   13   11     11400      9400    -15.32         1
   19    3   12      2000     10400     16.88         1
   20    5   12      4000     10400     15.87         1
   21    9   12      7400     10400      4.48         1
   22   11   12      9400     10400    -18.34         1
   23   13   13     11400     11400     -2.47         1
   24   14   13     13400     11400      8.10         1
   25    3   14      2000     12400     54.12         1
   26    5   14      4000     12400     38.27         1
   27   10   14      8400     12400      .053         1
   28   12   14     10400     12400     -2.92         1
   29    7   15      6000     13400      8.30         1
   30   14   15     12400     13400      4.54         1
   31    2   15      1000     14400     85.82         1
   32   11   15      9400     14400      2.26         1
    1    8    8    0    9
    2   10   10    0   11
    3   12   12    0   13
    4   10   10   14    0
    8         0
    9    .00012
   10        84
   11         0
   12         0
   13   .000051
   14      .008
    1   65.754   65.754         0 .00031149
    2   487.89   487.89         0-.00013995
    3   13.288   13.288         0 .00013516
    4   487.89   487.89   .080716         0
    1   16    7   16    1   80.978         0       .01
    7   16   15   16    2   935.15         0    .00056
   11   10   11   10    3   -97000      1940         1
   12    8   12    8    4   -50961      1020         1
IN       1    0
   15   15    5   16   -1    0
    1   10    5    6      1.04       .48
   15   16   10.198
   15   15
   15   14
   15   13
   15   12
   15   11
   15   10
   15    9
   15    8
   15    7   5.1211
    2    2    6    7       .48       .54
   15    6   5.1211
   15    5   5.4730
    4
```

Figure 1—Continued

```
 1     90.259
 2     1143.6
 3     -97085
 4     -50990
 5     10.313
 6     5.1349
 7     5.4858
 8     72.988
 9 3.4248E-4
10     490.46
11-1.8916E-4
12     22.688
13 2.2823E-4
14 8.0751E-2
 1     71.697
 2     726.73
 3     -96915
 4     -50932
 5     10.083
 6     5.1073
 7     5.4602
 8     58.520
 9 2.8050E-4
10     485.32
11-9.0735E-5
12     3.8882
13 4.2091E-5
14 8.0681E-2
 1     124.97
 2     1424.3
 3     -97222
 4     -51035
 5     10.476
 6     5.1538
 7     5.4698
 8     91.304
 9 3.6558E-4
10     494.12
11-2.5696E-4
12     15.271
13 1.5554E-4
14 8.0866E-2
 1     36.987
 2     446.01
 3     -96778
 4     -50887
 5     9.9203
 6     5.0884
 7     5.4762
 8     40.204
 9 2.5740E-4
10     481.66
11-2.2940E-5
12     11.305
13 1.1478E-4
14 8.0566E-2
 2     .98677
```

Problem 6.3-1

Compute

$$\hat{\gamma}=(\underline{Y}_p-\underline{X}_p\underline{\hat{b}}^*)^T[s^2\underline{X}_p(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T+\underline{U}]^{-1}\cdot(\underline{Y}_p-\underline{X}_p\underline{\hat{b}}^*).$$

For data set 1

$$\underline{\hat{b}}^*=\begin{bmatrix}50.018\\9.1954\\2.5008\times10^{-5}\end{bmatrix}.$$

For data set 2

$$\underline{\hat{b}}^*=\begin{bmatrix}50.055\\9.7914\\2.2766\times10^{-5}\end{bmatrix}.$$

For data set 1

$$(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}s^2=$$

$$\begin{bmatrix}2.5550\times10^{-1} & 8.9780\times10^{-2} & -1.7349\times10^{-6}\\ & 2.5550\times10^{-1} & -1.7349\times10^{-6}\\ \text{symmetric} & & 2.0715\times10^{-11}\end{bmatrix}$$

$$s^2\underline{X}_p(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T=1\times2.5550\times10^{-1}\times1$$

$$=0.25550$$

$$\underline{U}=(1.1)^2=1.21$$

$$s^2\underline{X}_p(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T+\underline{U}=0.25550+1.21$$

$$=1.46550$$

$$[s^2\underline{X}_p(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T+\underline{U}]^{-1}$$

$$=0.68236$$

$$\underline{Y}_p-\underline{X}_p\underline{\hat{b}}^*=11-9.1954=1.8046$$

$$\hat{\gamma}=1.8046\times0.68236\times1.8046=2.222$$

$$\chi^2_{0.05}(1)=3.841\ \therefore H_0 \text{ accepted.}$$

Prior and pure regression estimates of $\beta_2=h_b$ are in agreement.

For data set 2

$$(\underline{X}_s^T\underline{V}_s^{-1}\underline{X}_s)^{-1}s^2=$$

$$\begin{bmatrix}5.4121\times10^{-1} & 2.6265\times10^{-1} & -3.9795\times10^{-6}\\ & 5.4121\times10^{-1} & -3.9795\times10^{-6}\\ \text{symmetric} & & 4.3413\times10^{-11}\end{bmatrix}$$

$$s^2\underline{X}_p(\underline{X}_s\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T=1\times5.4121\times10^{-1}\times1$$

$$=0.54121$$

$$\underline{U}=(0.95)^2=0.9025$$

$$s^2\underline{X}_p(\underline{X}_s\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T+\underline{U}=0.54121+0.9025$$

$$=1.44371$$

$$[s^2\underline{X}_p(\underline{X}_s\underline{V}_s^{-1}\underline{X}_s)^{-1}\underline{X}_p^T+\underline{U}]^{-1}$$

$$=0.69266$$

$$\underline{Y}_p-\underline{X}_p\underline{\hat{b}}^*=9.5-9.7914=-0.2914$$

$$\hat{\gamma}=-0.2914\times0.69266\times(-0.2914)$$

$$=0.5882$$

$$\therefore H_0 \text{ accepted.}$$

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