Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B3

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

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Book 3
APPLICATIONS OF HYDRAULICS

Solution 7: Constant drawdown in a well in a leaky aquifer

Assumptions:

1. Water level in well is changed instantaneously by s_w at $t=0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\partial^2 s/\partial r^2 + (1/r)\partial s/\partial r - sK'/Tb' = (S/T)\partial s/\partial t$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic confined aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,0)=0,\ r\geq 0 \quad (1)$$

$$s(r_w,t)=s_w,\ t\geq 0 \quad (2)$$

$$s(\infty,t)=0,\ t\geq 0 \quad (3)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that at the wall or screen of the discharging well, drawdown in the aquifer is equal to the constant drawdown in the well, which assumes that there is no entrance loss to the discharging well. Equation 3 states that the drawdown approaches zero as distance from the discharging well approaches infinity.

Solutions (Hantush, 1959):

I. For the discharge rate of the well,

$$Q = 2\pi T s_w G(\alpha, r_w/B),$$

where

$$G(\alpha,r_w/B) = (r_w/B)K_1(r_w/B)/K_0(r_w/B) + (4/\pi^2)\exp\left[-\alpha(r_w/B)^2\right] \cdot \int_0^\infty \left\{u\exp(-\alpha u^2)/\left[J_0^2(u) + Y_0^2(u)\right]\right\} \cdot du/\left[u^2 + (r_w/B)^2\right],$$

and

$$\alpha = Tt/Sr_w^2,$$

$$B = \sqrt{Tb'/K'}.$$

K_0 and K_1 are zero-order and first-order, respectively, modified Bessel functions of the second kind. J_0 and Y_0 are the zero-order Bessel functions of the first and second kind, respectively.

II. For the drawdown in water level

$$s = s_w(K_0(r/B)/K_0(r_w/B) + (2/\pi)\exp(-\alpha r_w^2/B^2)\int_0^\infty \frac{\exp(-\alpha u^2)}{u^2 + (r_w/B)^2} \cdot \frac{J_0(ur/r_w)Y_0(u) - Y_0(ur/r_w)J_0(u)}{J_0^2(u) + Y_0^2(u)}\, u\, du \quad (4)$$

with α, B, K_0, J_0, and Y_0 as defined previously.

Comments:

A cross section through the discharging well is shown in figure 7.1. The boundary conditions most commonly apply to a flowing artesian well, as is shown in this illustration.

Figure 7.2 on plate 1 is a plot of dimensionless discharge ($G(\alpha,r_w/B)$) versus dimensionless time (α) from data of Hantush (1959, table 1) and Dudley (1970, table 2). Selected values of $G(\alpha,r_w/B)$ are given in table 7.1. The corresponding data curve should be a plot of observed discharge versus time. The data curve is matched to figure 7.2 and from match points ($\alpha,G(\alpha,r_w/B)$) and (t,Q), T and S are computed from the equations

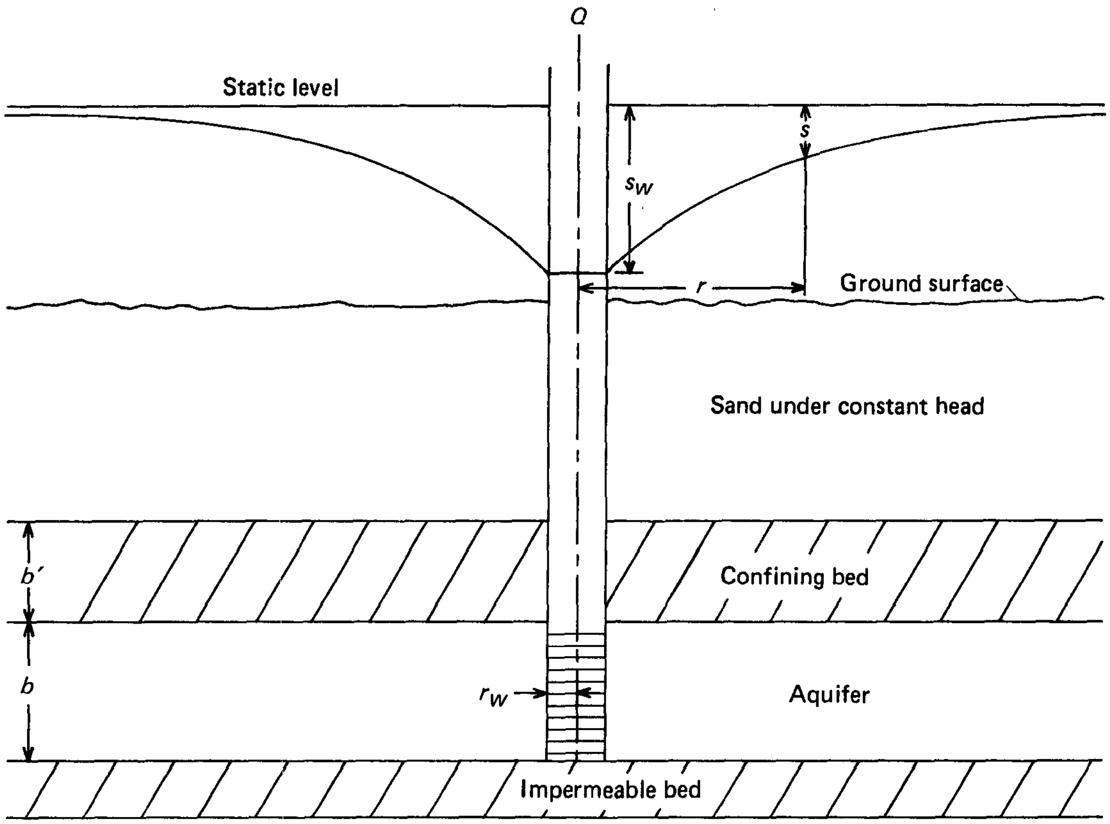

FIGURE 7.1.—Cross section through a well with constant drawdown in a leaky aquifer.

$$T = Q/(2\pi s_w G(\alpha, r_w/B))$$

and $$S = Tt/(\alpha r_w^2).$$

Figure 7.3 on plate 1 contains plots of dimensionless drawdown (s/s_w) versus dimensionless time ($\alpha r_w^2/r^2$). The corresponding data plot would be observed drawdown versus observation time. Matching the data and type curves by superposition and choosing convenient match points ($s/s_w, \alpha r_w^2/r^2$) and (s,t), the ratio of transmissivity to storage coefficient can be computed from the relation

$$T/S = (\alpha r_w^2/r^2)(r^2/t).$$

Figure 7.3 was plotted from function values generated by a FORTRAN program. This program is listed in table 7.2. The input data for this program consist of three cards coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of alpha for which computation is desired, coded in format E10.5 in columns 1–10; the largest value of alpha for which computation is desired, coded in format E10.5 in columns 11–20. The output table will include a range in alpha spanning these two values up to a limiting range of nine log cycles. The second card contains 13 values of r_w/B. These coded values are the significant figures only and should be greater or equal to 1 and less than 10. The power of 10 by which each of these coded values is multiplied is calculated by the program. Zero (or blank) coding is permissible, but the first zero (or blank) value will terminate the list. The 13 values, all coded in format F5.0, are coded in columns 1–5, 6–10, 11–15, 16–20, 21–25, 26–30, 31–35, 36–40, 41–45, 46–50, 51–55, 56–60, and 61–65. The third card contains the radius of the control well and distances to the observation wells.

TABLE 7.1.—*Values of* $G(\alpha, r_w/B)$

[Values for $r_w/B \leqslant 1\times10^{-2}$ and $\alpha \geqslant 1\times10^{2}$ are from Hantush (1959, table 1), others are from Dudley (1970, table 2)]

α	r_w/B: 0	6×10^{-3}	1×10^{-2}	2×10^{-2}	6×10^{-2}	1×10^{-1}	2×10^{-1}	6×10^{-1}	1×10^{0}
1×10^{-1}	2.24	2.24	2.24	2.25	2.25	2.25	2.26	2.31	2.43
2	1.71	1.71	1.71	1.71	1.72	1.72	1.73	1.81	1.96
5	1.23	1.23	1.23	1.23	1.23	1.24	1.25	1.38	1.61
1×10^{0}	.983	.983	.983	.984	.986	.990	1.01	1.18	1.49
2	.800	.800	.800	.801	.804	.809	.834	1.07	1.44
5	.628	.628	.628	.629	.633	.642	.682	1.01	1.43
1×10^{1}	.534	.534	.534	.535	.541	.554	.611		
2	.461	.461	.461	.462	.472	.491	.569		
5	.389	.389	.389	.390	.407	.438	.548		
1×10^{2}	.346	.346	.346	.349	.374	.417	.545		
2	.311	.311	.312	.316	.353	.408			
5	.274	.275	.276	.284	.341	.406			
1×10^{3}	.251	.252	.255	.266	.339				
2	.232	.234	.239	.255					
5	.210	.215	.222	.249					
1×10^{4}	.196	.204	.216	.248					
2	.185	.197	.213						
5	.170	.192	.212						
1×10^{5}	.161	.191							
2	.152								
5	.143								
1×10^{6}	.136								
2	.130								
5	.123	.191	.212	.248	.339	.406	.545	1.01	1.43

α	r_w/B: 0	1×10^{-5}	2×10^{-5}	6×10^{-5}	1×10^{-4}	2×10^{-4}	6×10^{-4}	1×10^{-3}	2×10^{-3}
1×10^{4}	0.196	0.196	0.196	0.196	0.196	0.196	0.196	0.196	0.197
2	.185	.185	.185	.185	.185	.185	.185	.185	.185
5	.170	.170	.170	.170	.170	.170	.170	.170	.173
1×10^{5}	.161	.161	.161	.161	.161	.161	.162	.162	.167
2	.152	.152	.152	.152	.152	.152	.153	.155	.163
5	.143	.143	.143	.143	.143	.143	.144	.148	.161
1×10^{6}	.136	.136	.136	.136	.136	.137	.139	.144	.159
2	.130	.130	.130	.130	.130	.131	.135	.143	.159
5	.123	.123	.123	.123	.123	.124	.133	.142	.158
1×10^{7}	.118	.118	.118	.118	.118	.120			
2	.114	.114	.114	.114	.114	.116			
5	.108	.108	.108	.108	.110				
1×10^{8}	.104	.104	.104	.105	.108				
2	.100	.100	.101	.103	.107				
5	.0958	.0958	.0966	.102					
1×10^{9}	.0927	.0930	.0943						
2	.0899	.0906	.0927						
5	.0864	.0880	.0916						
1×10^{10}	.0838	.0867	.0914						
2	.0814	.0862							
5	.0785	.0860							
1×10^{11}	.0764	.0860	.0914	.102	.107	.116	.133	.142	.158
2									
5									

The control well radius (r_w) is coded first, in columns 1–8 in format F8.2. The distances (r) to the observation wells (maximum of nine) are coded next, in monotonic increasing order (smallest r first, largest r last), in columns 9–16, 17–24, 25–32, 33–40, 41–48, 49–56, 57–64, 65–72, and 73–80, all in format F8.2. If two or more observation wells have the same distance, this common distance should be coded only once, the function values will apply to all wells at the same distance from the control well. If the number of observation wells is less than nine, the remaining columns on the card should be left blank.

The integral in equation 4 is approximated by

$$\int_0^\infty f(u,\alpha,r_w/B)\,du \doteq \sum_{i=1}^{8000} f(-\Delta u/2 + i\Delta u,\alpha,r_w/B)\,\Delta u\,.$$

This expression is a composite quadrature with equally spaced abscissas. The abscissas are chosen at the midpoints of the intervals instead of the ends because the integrand is singular at $u=0$. The value of Δu used is related to α and is $\Delta u \leq 10^{-3}/\sqrt{\alpha}$. The r_w/B values then selected by the program satisfy $r_w/B \geq 10\ \Delta u$. These two constraints, though empirical, are related to the behavior of the integrand; the first constraint is related to the term $e^{-\alpha u^2}$ as u becomes large, and the second to $u/(u^2+(r_w/B)^2)$ as u becomes small.

The Bessel functions $K_0(r/B)$, $K_0(r_w/B)$ are evaluated by the IBM subroutine BESK. A description of this subroutine may be found in the IBM Scientific Subroutine Package.

The Bessel functions of the second kind in the integrand, $Y_0(u)$ and $Y_0(ur/r_w)$, are evaluated using IBM subroutine BESY, which is discussed in IBM SSP manual. The Bessel functions $J_0(u)$ and $J_0(ur/r_w)$ are evaluated for arguments less than four by a polynomial approximation consisting of the first 10 terms of the series expansion

$$J_0(x) = \sum_{u=0}^{\infty} (-1)^u (x^2/2)^u/(n!)^2.$$

For arguments greater than or equal to four, the asymptotic expansion is used

$$J_0(x) = P \cos (x - \pi/4) + Q \sin (x - \pi/4).$$

P and Q are calculated by the algorithm used in IBM subroutine BESY.

The output from this program consists of tables of function values, an example of which is shown in figure 7.4.

Solution 8: Constant discharge from a fully penetrating well of finite diameter in a nonleaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived from a depletion of storage in the aquifer and inside the well bore.

Differential equation:

$$\partial^2 s/\partial r^2+(1/r)\partial s/\partial r = (S/T)\partial s/\partial t,\ r\geq r_w$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer in the region outside the pumped well.

Boundary and initial conditions:

$$s(r_w, t) = s_w(t),\ t>0 \qquad (1)$$

$$s(\infty,t)=0,\ t>0 \qquad (2)$$

$$s(r, 0) = 0,\ r\geq r_w \qquad (3)$$

$$s_w(0) = 0 \qquad (4)$$

$$(2\pi r_w T)\partial s(r_w, t)/\partial r-(\pi r_c^2)\partial s_w(t)/\partial t = -Q,\ t>0 \qquad (5)$$

Equation 1 states that the drawdown at the well bore is equal to the drawdown inside the well, assuming that there is no entrance loss at the well face. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that, initially, drawdown in the aquifer and inside the well is zero. Equation 5 states that the discharge of the well is equal to the sum of the flow into the well and the rate of decrease in storage inside the well.

Solution (Papadopulos and Cooper, 1967; Papadopulos, 1967):

$$s = (Q/4\pi T)\,F(u,\alpha,\rho),$$

where

$$F(u,\alpha,\rho) = (8\alpha/\pi)\int_0^{\infty} \frac{[(1-\exp(-\beta^2\rho^2/4u)]\ [J_0(\beta\rho)A(\beta)-Y_0(\beta\rho)B(\beta)]}{\{[A(\beta)]^2 + [B(\beta)]^2\}\beta^2}\,d\beta,$$

and

$$B(\beta) = \beta J_0(\beta)-2\alpha J_1(\beta),$$
$$A(\beta) = \beta Y_0(\beta)-2\alpha Y_1(\beta),$$
$$u = r^2S/4Tt,$$
$$\alpha = r_w^2S/r_c^2,$$

and

$$\rho = r/r_w.$$

J_0 and Y_0, J_1 and Y_1, are zero-order and first-order Bessel functions of the first and second kind, respectively.

Z(ALPHA,R/RW,RW/B), R/RW= 100.

ALPHA | RW/B	0.10E-03	0.15E-03	0.20E-03	0.30E-03	0.50E-03	0.70E-03	0.10E-02	0.15E-02	0.20E-02	0.30E-02	0.50E-02	0.70E-02	0.10E-01
0.100E 05	0.114	0.114	0.114	0.114	0.113	0.113	0.113	0.112	0.112	0.109	0.102	0.091	0.074
0.150E 05	0.142	0.142	0.142	0.141	0.141	0.141	0.141	0.140	0.138	0.134	0.122	0.107	0.082
0.200E 05	0.161	0.161	0.161	0.161	0.161	0.161	0.160	0.159	0.157	0.151	0.135	0.115	0.086
0.300E 05	0.188	0.188	0.188	0.188	0.188	0.188	0.187	0.184	0.181	0.173	0.150	0.123	0.088
0.500E 05	0.221	0.221	0.221	0.221	0.220	0.220	0.218	0.214	0.209	0.196	0.162	0.128	0.089
0.700E 05	0.242	0.242	0.242	0.241	0.241	0.240	0.237	0.232	0.225	0.208	0.167	0.130	0.089
0.100E 06	0.263	0.262	0.262	0.262	0.261	0.260	0.257	0.250	0.240	0.218	0.169	0.130	0.089
0.150E 06	0.285	0.285	0.285	0.284	0.283	0.281	0.277	0.267	0.254	0.225	0.170	0.130	0.089
0.200E 06	0.300	0.300	0.300	0.299	0.298	0.295	0.289	0.277	0.262	0.228	0.171	0.130	0.089
0.300E 06	0.321	0.321	0.320	0.319	0.317	0.313	0.305	0.289	0.269	0.231	0.171	0.130	0.089
0.500E 06	0.345	0.345	0.344	0.343	0.339	0.333	0.322	0.299	0.275	0.232	0.171	0.130	0.089
0.700E 06	0.360	0.360	0.359	0.357	0.352	0.344	0.330	0.303	0.276	0.232	0.171	0.130	0.089
0.100E 07	0.375	0.375	0.374	0.371	0.364	0.355	0.337	0.305	0.277	0.232	0.171	0.130	0.089
0.150E 07	0.391	0.391	0.389	0.386	0.376	0.364	0.342	0.306	0.277	0.232	0.171	0.130	0.089
0.200E 07	0.402	0.401	0.400	0.396	0.384	0.368	0.344	0.307	0.277	0.232	0.171	0.130	0.089
0.300E 07	0.417	0.416	0.414	0.408	0.392	0.373	0.345	0.307	0.277	0.232	0.171	0.130	0.089
0.500E 07	0.435	0.432	0.429	0.421	0.399	0.376	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.700E 07	0.445	0.442	0.438	0.427	0.401	0.376	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.100E 08	0.456	0.452	0.446	0.43[illegible]	0.403	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.150E 08	0.467	0.461	0.454	0.437	0.403	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.200E 08	0.474	0.467	0.458	0.439	0.404	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.300E 08	0.483	0.473	0.462	0.440	0.404	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.500E 08	0.492	0.479	0.465	0.440	0.404	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.700E 08	0.497	0.482	0.466	0.440	0.404	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089
0.100E 09	0.501	0.483	0.467	0.440	0.404	0.377	0.346	0.307	0.277	0.232	0.171	0.130	0.089

FIGURE 7.4.—Example of output from program for constant drawdown in a well in a leaky artesian aquifer.

The drawdown inside the pumped well is obtained at $r = r_w$ and can be expressed as (Papadopulos and Cooper, 1967, p. 242):

$$s_w = (Q/4\pi T)\,F(u_w,\alpha),$$

where $$F(u_w,\alpha) = F(u,\alpha,1),$$

and $$u_w = r_w^2 S/4tT.$$

Comments: A cross section through the discharging well is shown in figure 8.1. The geometry, except for the region of the well bore, is the same as for solution 1 (Theis solution). It is apparent from figure 8.2 and 8.3 (on plate 1) that $F(u,\alpha,\rho)$ approaches $W(u)$, the Theis solution, as time becomes large. Papadopulos (1967, p. 161) stated that for $t>2.5\times10^3 r_c/T$, or $\alpha\rho^2/u>10^4$, the function $F(u,\alpha,\rho)$ can be closely approximated by $F(u,\alpha,\rho)=W(u)$. Papadopulos and Cooper (1967, p. 242) stated that for $t>2.5\times10^2\ r_c^2/T$, or $\alpha/u_w>10^3$, the function $F(u_w,\alpha)$ can be closely approximated by $F(u_w,\alpha)=W(u_w)$. An examination of the type curves and function values indicates that $F(u_w,\alpha)\approx W(u_w)$ (less than 5-percent error) for $\alpha/u_w>10^2$, and hence t should only be greater than 25 r_c^2/T for drawdown in the pumped well.

Figures 8.2 and 8.3 were prepared from function values given in Papadopulos and Cooper (1967) and Papadopulos (1967), which are reproduced in table 8.1. For drawdown observations in the pumped well, the method of analysis is to plot drawdown versus time and

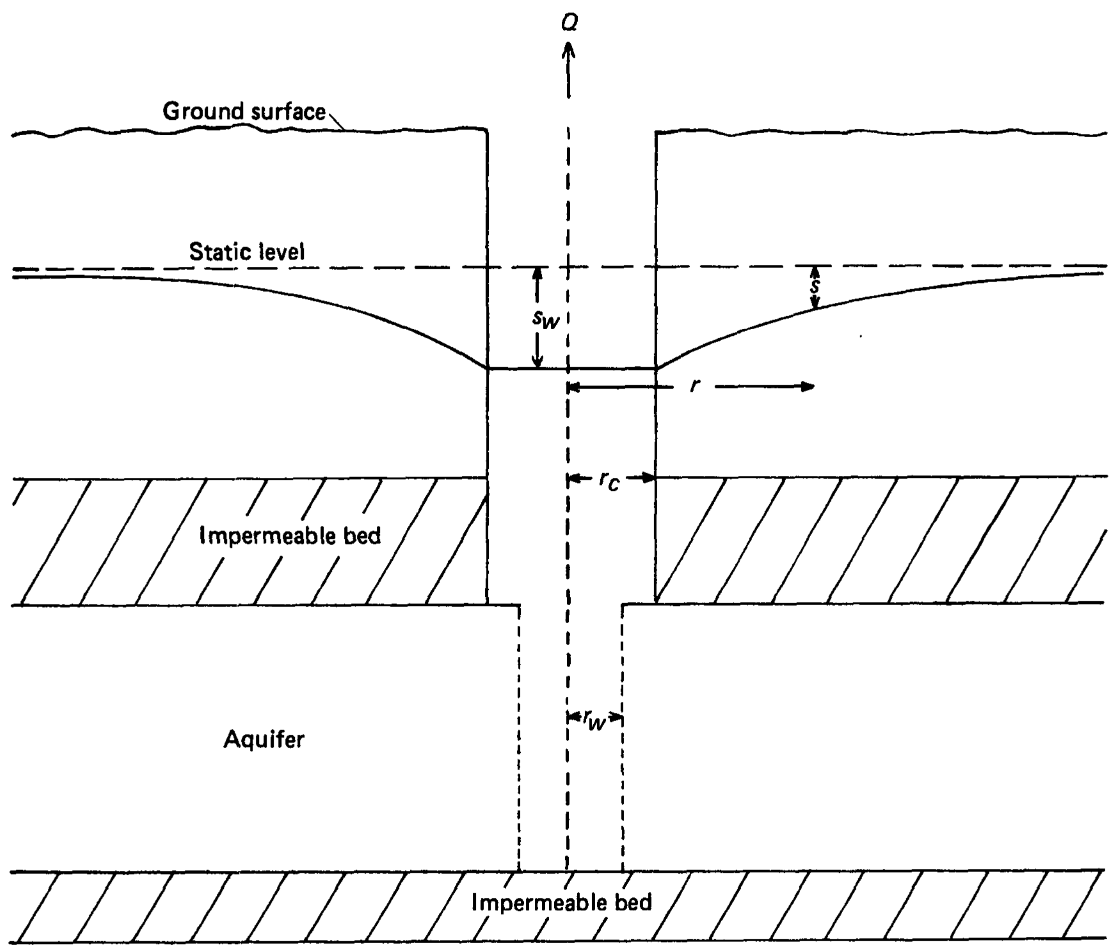

FIGURE 8.1.—Cross section through a discharging well of finite diameter.

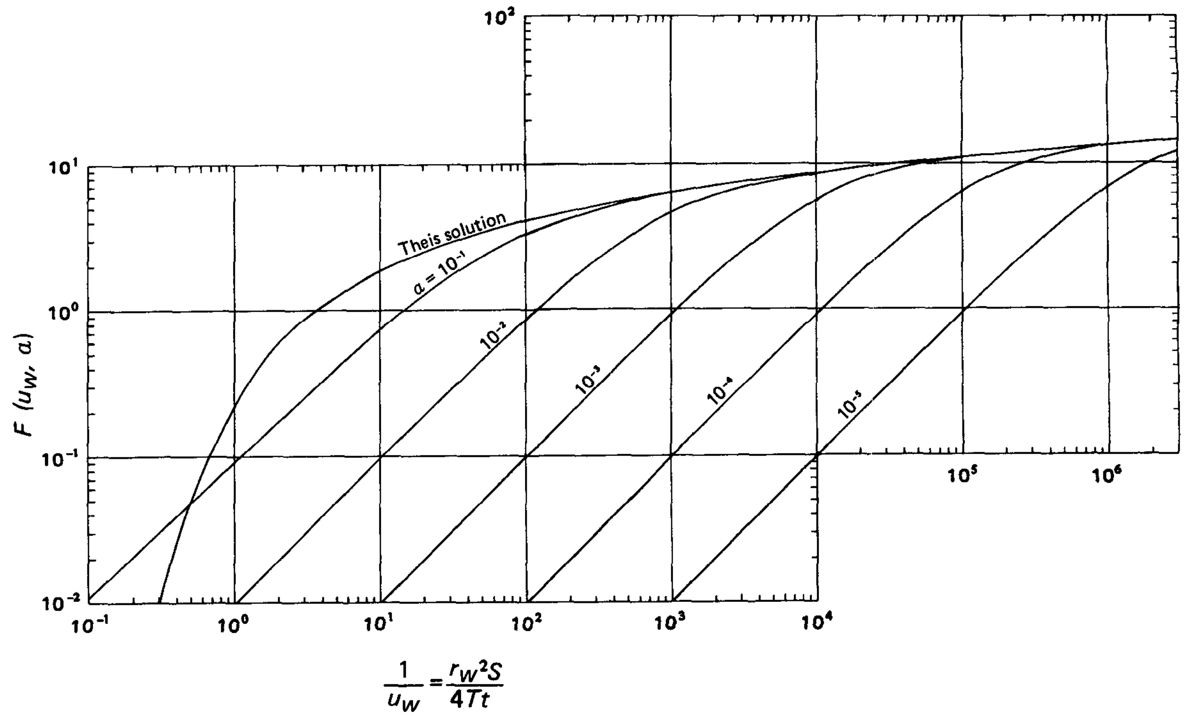

FIGURE 8.2.—Five selected type curves of $F(u_w,\alpha)$, and the Theis solution, versus $1/u_w$.

then superimpose the plot on figure 8.2. After match points of (s,t) and $(F(u_w,\alpha),\ 1/u_w)$ are chosen, the transmissivity can be computed from the relation $T=(Q/4\pi s)\ F(u_w,\alpha)$. Then, the storage coefficient can be determined from $S=(4\,Tt/r_w^2)/(1/u_w)$.

For observations not in the pumped well, two procedures are available for analyzing the data. To analyze the data from a single observation well, a family of type curves of $F(u,\alpha,\rho)$ versus $1/u$ for different values of α can be plotted for the ρ value appropriate for the observation well, using values in table 8.1. This procedure produces a family of type curves similar to that shown for $\rho=1$ in figure 8.2. If ρ for the observation well is between ρ values in table 8.1, function values can be interpolated. Using this approach, the data for the observation well are plotted as drawdown versus time and matched to the best-fitting member of the plotted type curves. Transmissivity and storage coefficient can be calculated from $T=(Q/4\pi s)$ $F(u,\alpha,\rho)$ and $S=(4Tt/r^2)/(1/u)$.

Drawdowns at more than one observation point may be combined by preparing a composite plot of the drawdowns at each observation well versus t/r^2. This composite plot would be analyzed by matching it to a family of type curves of $F(u,\alpha,\rho)$ versus $1/u$ for constant α. An example of such a type-curve family for $\alpha=10^{-4}$ is shown in figure 8.3. This method requires multiple sheets of type curves, one sheet for each value of α. When the data curves are matched to the type-curve family, care should be taken to insure that the data for each well fall on the type curve having the appropriate ρ value. This will be possible for all the data for only one value of α. Transmissivity and storage coefficient are calculated from $T=(Q/4\pi s)$ $F(u,\alpha,\rho)$ and $S=4T(t/r^2)/(1/u)$.

In both of these methods of plotting and comparing data, an alternate computation of storage coefficient is $S=r_c^2\alpha/r_w^2$. However, as pointed out by Papadopulos and Cooper (1967, p. 244), the shapes of type curves differ only slightly when α changes by an order of magnitude, therefore the determination of S is sensitive to choosing the "correct" curve. Papadopulos and Cooper (1967, p. 244) suggest that if S can be estimated within an order of magnitude, the value of α to be used for matching the data can be decided.

TABLE 8.1.—*Values of the function* $F(u,\alpha,\rho)$

[Values for ρ = 1 from Papadopulos and Cooper, 1967. Other values from Papadopulos, 1967]

u	ρ = 1	2	5	10	20	50	100	200
For $\alpha = 10^{-1}$								
2×10^{0}	4.88×10^{-2}	1.96×10^{-2}	1.75×10^{-2}	2.41×10^{-2}	3.48×10^{-2}	4.24×10^{-2}	4.48×10^{-2}	4.50×10^{-2}
1	9.19	7.01	9.55	1.41×10^{-1}	1.85×10^{-1}	2.09×10^{-1}	2.14×10^{-1}	2.15×10^{-1}
5×10^{-1}	1.77×10^{-1}	1.95×10^{-1}	3.21×10^{-1}	4.44	5.20	5.49	5.55	5.59
2	4.06	5.78	9.42	1.13×10^{0}	1.19×10^{0}	1.22×10^{0}		
1	7.34	1.11×10^{0}	1.60×10^{0}	1.76	1.80			
5×10^{-2}	1.26×10^{0}	1.84	2.33	2.43	2.46			
2	2.30	2.97	3.28	3.34	3.35			
1	3.28	3.81	4.00	4.03				
5×10^{-3}	4.26	4.60	4.70	4.72				
2	5.42	5.58	5.63	5.64				
1	6.21	6.30	6.33					
5×10^{-4}	6.96	7.01						
2	7.87	7.93						
1	8.57	8.63						
5×10^{-5}	9.32							
2	10.24							
For $\alpha = 10^{-2}$								
2×10^{0}	4.99×10^{-3}	2.13×10^{-3}	2.11×10^{-3}	3.52×10^{-3}	7.47×10^{-3}	2.03×10^{-2}	3.44×10^{-2}	4.35×10^{-2}
1	9.91	7.99	1.32×10^{-2}	2.69×10^{-2}	6.12×10^{-2}	1.42×10^{-1}	1.91×10^{-1}	2.11×10^{-1}
5×10^{-1}	1.97×10^{-2}	2.40×10^{-2}	5.40	1.21×10^{-1}	2.63×10^{-1}	4.65	5.31	5.51
2	4.89	8.34	2.33×10^{-1}	5.12	9.15	1.16×10^{0}	1.20×10^{0}	1.22×10^{0}
1	9.67	1.93×10^{-1}	5.67	1.12×10^{0}	1.58×10^{0}	1.78	1.81	
5×10^{-2}	1.90×10^{-1}	4.16	1.18×10^{0}	1.95	2.32	2.44	2.46	
2	4.53	1.03×10^{0}	2.42	3.11	3.29	3.34	3.35	
1	8.52	1.87	3.48	3.90	4.00	4.03		
5×10^{-3}	1.54×10^{0}	3.05	4.43	4.65	4.71	4.72		
2	3.04	4.78	5.52	5.61	5.63	5.64		
1	4.55	5.90	6.27	6.31	6.33			
5×10^{-4}	6.03	6.81	6.99	7.01				
2	7.56	7.85	7.92	7.94				
1	8.44	8.59	8.63					
5×10^{-5}	9.23	9.30						
2	10.20	10.23						
1	10.87	10.93						
5×10^{-6}	11.62	11.63						
2	12.54							
1	13.24							

TABLE 8.1.—*Values of the function* $F(u,\alpha,\rho)$—Continued

u	ρ = 1	2	5	10	20	50	100	200
				For $\alpha = 10^{-3}$				
2×10^{0}	5.00×10^{-4}	2.15×10^{-4}	2.15×10^{-4}	3.70×10^{-4}	8.35×10^{-3}	3.05×10^{-3}	8.38×10^{-3}	1.50×10^{-2}
1	9.99	8.11	1.37×10^{-3}	2.95×10^{-3}	7.58×10^{-3}	2.81×10^{-2}	7.56×10^{-2}	1.47×10^{-1}
5×10^{-1}	2.00×10^{-3}	2.45×10^{-3}	5.77	1.42×10^{-2}	3.90×10^{-2}	$1.54 \times {}^{-1}$	3.23×10^{-1}	4.78
2	4.99	8.71	2.67×10^{-2}	7.24	2.03×10^{-1}	6.59	1.02×10^{0}	1.17×10^{0}
1	9.97	2.07×10^{-2}	7.16	2.01×10^{-1}	5.41	1.38×10^{0}	1.70	1.79
5×10^{-2}	1.99×10^{-2}	4.66	1.74×10^{-1}	4.87	1.19×10^{0}	2.27	2.40	2.45
2	4.95	1.29×10^{-1}	5.05	1.31×10^{0}	2.52	3.22	3.32	3.35
1	9.83	2.70	1.04×10^{0}	2.38	3.59	3.96	4.02	
5×10^{-3}	1.95×10^{-1}	5.47	1.96	3.68	4.50	4.69	4.72	
2	4.73	1.31×10^{0}	3.81	5.23	5.55	5.63	5.64	
1	9.07	2.39	5.34	6.13	6.28	6.32		
5×10^{-4}	1.69×10^{0}	3.98	6.57	6.92	7.00	7.02		
2	3.52	6.44	7.77	7.90	7.93			
1	5.53	7.95	8.55	8.61	8.63			
5×10^{-5}	7.63	9.02	9.28	9.31				
2	9.68	10.12	10.22	10.24				
1	10.68	10.88	10.93					
5×10^{-6}	11.50	11.59	11.62					
2	12.49	12.53	12.54					
1	13.21	13.23	13.24					
5×10^{-7}	13.92	13.93						
2	14.84							
1	15.54							
				For $\alpha = 10^{-4}$				
2×10^{0}	5.00×10^{-5}	2.17×10^{-5}	2.18×10^{-5}	3.73×10^{-5}	8.46×10^{-5}	3.16×10^{-4}	9.56×10^{-4}	3.83×10^{-3}
1	1.00×10^{-4}	8.15	1.38×10^{-4}	2.98×10^{-4}	7.77×10^{-4}	3.23×10^{-3}	1.01×10^{-2}	3.42×10^{-2}
5×10^{-1}	2.00	2.47×10^{-4}	5.81	1.45×10^{-3}	4.10×10^{-3}	1.80×10^{-2}	5.62	1.75×10^{-1}
2	5.00	8.76	2.71×10^{-3}	7.54	2.27×10^{-2}	1.03×10^{-1}	3.04×10^{-1}	7.10
1	1.00×10^{-3}	2.09×10^{-3}	7.34	2.16×10^{-2}	6.69	2.97	7.92	1.43×10^{0}
5×10^{-2}	2.00	4.72	1.82×10^{-2}	5.55	1.74×10^{-1}	7.30	1.62×10^{0}	2.24
2	5.00	1.32×10^{-2}	5.56	1.74×10^{-1}	5.36	1.87×10^{-0}	2.95	3.28
1	9.98	2.81	1.23×10^{-1}	3.86	1.14×10^{0}	3.08	3.84	4.02
5×10^{-3}	1.99×10^{-2}	5.88	2.64	8.13	2.17	4.25	4.63	4.71
2	4.97	1.53×10^{-1}	6.89	1.97×10^{0}	4.14	5.47	5.60	5.63
1	9.90	3.10	1.36×10^{0}	3.44	5.61	6.24	6.31	6.33
5×10^{-4}	1.97×10^{-1}	6.18	2.53	5.26	6.71	6.98	7.01	
2	4.81	1.48×10^{0}	4.95	7.33	7.82	7.92	7.94	
1	9.34	2.72	7.03	8.37	8.57	8.62		
5×10^{-5}	1.77×10^{0}	4.65	8.65	9.20	9.29	9.32		
2	3.83	7.87	10.02	10.19	10.23	10.24		
1	6.25	9.92	10.83	10.91	10.93			
5×10^{-6}	8.99	11.23	11.57	11.62	11.63			

2	11.74	12.40	12.52	12.54				
1	12.91	13.17	13.23	13.24				
5×10^{-7}	13.78	13.90	13.93					
2	14.79	14.83						
1	15.51	15.53						
5×10^{-8}	16.22	16.23						
2	17.14							
1	17.84							
For $\alpha = 10^{-5}$								
2×10^{0}	5.00×10^{-6}	2.27×10^{-6}	2.48×10^{-6}	4.19×10^{-6}	9.00×10^{-6}	3.21×10^{-5}	9.77×10^{-5}	3.15×10^{-4}
1	1.00×10^{-5}	8.36	1.44×10^{-5}	3.07×10^{-5}	7.89×10^{-5}	3.27×10^{-4}	1.04×10^{-3}	3.44×10^{-3}
5×10^{-1}	2.00	2.51×10^{-5}	5.94	1.47×10^{-4}	4.14×10^{-4}	1.84×10^{-3}	6.02	2.00×10^{-2}
2	5.00	8.87	2.74×10^{-4}	7.61	2.31×10^{-3}	1.08×10^{-2}	3.61×10^{-2}	1.19×10^{-1}
1	1.00×10^{-4}	2.11×10^{-4}	7.42	2.18×10^{-3}	6.85	3.30	1.10×10^{-1}	3.50
5×10^{-2}	2.00	4.77	1.84×10^{-3}	5.65	1.82×10^{-2}	8.90	2.92	8.57
2	5.00	1.34×10^{-3}	5.64	1.80×10^{-2}	5.92	2.89×10^{-1}	8.91	2.12×10^{0}
1	1.00×10^{-3}	2.84	1.26×10^{-2}	4.09	1.36×10^{-1}	6.49	1.80×10^{0}	3.34
5×10^{-3}	2.00	5.96	2.74	9.03	3.01	1.35×10^{0}	3.14	4.40
2	5.00	1.56×10^{-2}	7.43	2.47×10^{-1}	8.06	3.03	5.01	5.52
1	9.99	3.20	1.55×10^{-1}	5.15	1.60×10^{0}	4.75	6.06	6.27
5×10^{-4}	2.00×10^{-2}	6.54	3.20	1.04×10^{0}	2.96	6.31	6.90	6.99
2	4.98	1.66×10^{-1}	8.08	2.45	5.58	7.71	7.89	7.93
1	9.93	3.34	1.58×10^{0}	4.28	7.54	8.52	8.61	8.63
5×10^{-5}	1.98×10^{-1}	6.62	2.93	6.63	8.90	9.21	9.31	
2	4.86	1.59×10^{0}	5.86	9.36	10.10	10.22	10.24	
1	9.49	2.95	8.53	10.60	10.86	10.92		
5×10^{-6}	1.82×10^{0}	5.15	10.67	11.48	11.59	11.62		
2	4.03	9.08	12.28	12.49	12.53	12.54		
1	6.78	11.76	13.12	13.21	13.23	13.24		
5×10^{-7}	10.13	13.41	13.88	13.92	13.93			
2	13.71	14.68	14.83	14.85				
1	15.13	15.46	15.54					
5×10^{-8}	16.05	16.20						
2	17.08	17.14						
1	17.81	17.84						
5×10^{-9}	18.51							
2	19.40							
1	20.15							

The early parts (short time) of the curves in figure 8.2 are straight lines. According to Papadopulos and Cooper (1967, p. 244), these represent conditions under which all the water pumped is derived from storage within the well. The straight lines approached by the curves satisfy the equations

$$F(u_w,\alpha) = \alpha/u_w$$

and

$$s_w = Qt/\pi r_c^2 = \frac{\text{volume of water discharged}}{\text{area of well}}$$

Therefore, as pointed out by Papadopulos and Cooper (1067, p. 244), data that fall on this straight part of the type curves do not indicate information about the aquifer characteristics.

Table 8.2 is a listing of two FORTRAN programs by S. S. Papadopulos that evaluate $F(u_w,\alpha)$ and $F(u,\alpha,\rho)$. The input data to both programs consists of cards coded in specified format (readers unfamiliar with FORTRAN language format should refer to a FORTRAN language manual). Input to the programs is one or more groups of data, each group of data consisting of two cards. The first card contains one value of alpha in columns 1–10, coded in format E10.5. The program to evaluate $F(u,\alpha,\rho)$ also requires a value of rho on this card in columns 11–20. This value of rho, which must be greater than one, is also coded in format E10.5. The second card contains 16 values of u coded in columns 1–5, 6–10, . . . , 75–80 in format 16F5.0. The $F(u_w,\alpha)$ or $F(u,\alpha,\rho)$ values will be printed in the order that the u values are coded. If less than 16 values of u are desired, the remaining columns on the card may be left blank. Outputs from these two programs are shown in figures 8.4 and 8.5.

F(UW,ALPHA) FOR ALPHA= 1.00000E-04

UW	INTEGRAL	INTEGRAL ERROR	F(UW,ALPHA)	X(PEAK)	Y(PEAK)
2.00000E 00	1.54210E 03	-6.98844E-02	4.99991E-05	5.96561E-03	5.55886E 05
1.00000E 00	3.08412E 03	-1.39817E-01	9.99956E-05	5.96561E-03	1.11177E 06
5.00000E-01	6.16789E 03	-2.74775E-01	1.99980E-04	5.96561E-03	2.22353E 06
2.00000E-01	1.54184E 04	-6.97533E-01	4.99907E-04	5.96561E-03	5.55875E 06
1.00000E-01	3.08331E 04	-1.39715E 00	9.99695E-04	5.96560E-03	1.11173E 07
5.00000E-02	6.16529E 04	-2.71364E 00	1.99896E-03	5.96559E-03	2.22335E 07
2.00000E-02	1.54061E 05	-6.97112E 00	4.99507E-03	5.96559E-03	5.55764E 07
1.00000E-02	3.07919E 05	-1.39383E 01	9.98359E-03	5.96554E-03	1.11128E 08
5.00000E-03	6.15138E 05	-2.78767E 01	1.99445E-02	5.96549E-03	2.22157E 08
2.00000E-03	1.53334E 06	-6.82757E 01	4.97152E-02	5.96527E-03	5.54652E 08
1.00000E-03	3.05367E 06	-1.38658E 02	9.90083E-02	5.96493E-03	1.10684E 09
5.00000E-04	6.06085E 06	-2.76458E 02	1.96509E-01	5.96425E-03	2.20389E 09
2.00000E-04	1.48475E 07	-6.79220E 02	4.81397E-01	5.96223E-03	5.43712E 09
1.00000E-04	2.88072E 07	-1.30780E 03	9.34008E-01	5.95886E-03	1.06380E 10
5.00000E-05	5.45352E 07	-2.50960E 03	1.76818E 00	5.95237E-03	2.03734E 10
2.00000E-05	1.18065E 08	-5.40026E 03	3.82800E 00	5.93415E-03	4.49196E 10

FIGURE 8.4.—Example of output from program for drawdown inside a well of finite diameter due to constant discharge.

F(U,ALPHA,RHO) FOR ALPHA= 1.00000E-05, RHO= 2.00000E 00

U	INTEGRAL	INTEGRAL ERROR	F(U,ALPHA,RHO)
9.99999900E-04	6.29273600E 02	5.45096700E-01	3.20486300E-02
5.00000000E-04	1.28359500E 03	1.11649700E 00	6.53728800E-02
1.99999900E-04	3.26376700E 03	2.47402200E 00	1.66222200E-01
1.00000000E-04	6.55423000E 03	3.31468400E 00	3.33803700E-01
5.00000000E-05	1.30015800E 04	3.53750700E 00	6.62164900E-01
2.00000000E-05	3.11692500E 04	3.54940500E 00	1.58743500E 00
9.99999900E-06	5.79505700E 04	3.54602200E 00	2.95139600E 00
4.99999900E-06	1.01023500E 05	3.53222000E 00	5.14508300E 00
1.99999900E-06	1.78237100E 05	3.62180400E 00	9.07753300E 00
1.00000000E-06	2.30897600E 05	3.66347000E 00	1.17595100E 01
4.99999900E-07	2.63222100E 05	3.68847000E 00	1.34057800E 01
1.99999900E-07	2.88201800E 05	3.52180300E 00	1.46779900E 01

FIGURE 8.5.—Example of output from program for drawdown outside a well of finite diameter due to constant discharge.

Solution 9: Slug test for a finite-diameter well in a nonleaky aquifer

Assumptions:

1. A volume of water, V, is injected into, or is discharged from, the well instantaneously at $t=0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky, and flow is in radial direction only.

Differential equation:

$$\partial^2 h/\partial r^2 + (1/r)\,\partial h/\partial r = (S/T)\,\partial h/\partial t,\ r>r_w$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer beyond the radius of the injected well.

Boundary and initial conditions:

$$h(r_w,t)=H(t),\ t>0 \quad (1)$$
$$h(\infty,t)=0,\ t>0 \quad (2)$$
$$2\pi r_w T\,\frac{\partial h(r_w,t)}{\partial r} = \pi r_c^2\,\frac{\partial H(t)}{\partial t},\ t>0 \quad (3)$$
$$h(r,0)=0,\ r>r_w \quad (4)$$
$$H(0) = H_0 = V/\pi r_c^2 \quad (5)$$

Equation 1 states that the head change in the aquifer at the face of the well is equal to that inside the well; one assumes that there is no exit loss at the well face. Equation 2 states that the head change approaches zero as distance from the discharging well approaches infinity, a condition which will be approximated if boundaries of the aquifer are sufficiently distant from the discharging well. Equation 3 states that near the well the radial flow is equal to the rate of change in volume of water inside the well. Equations 4 and 5 state that initially the head change is zero in the aquifer, and the head increase or decrease inside the well is equal to H_0.

Solution (Cooper and others, 1967):

$$h = (2H_0/\pi)\int_0^{\infty} \Big(\exp(-\beta u^2/\alpha)\big\{ J_0(ur/r_w)\cdot\left[uY_0(u)- 2\alpha Y_1(u)\right] - Y_0(ur/r_w)\cdot\left[uJ_0(u)- 2\alpha J_1(u)\right]\big\} /\Delta(u)\Big)\, du, \quad (6)$$

where $\alpha = r_w^2 S/r_c^2$,
$\beta = Tt/r_c^2$,

and
$$\Delta(u) = \left[uJ_0(u)-2\alpha J_1(u)\right]^2 + \left[uY_0(u)-2\alpha Y_1(u)\right]^2.$$

J_0 and Y_0, J_1 and Y_1, are zero-order and first-order Bessel functions of the first and second kind, respectively.

The head, H, inside the well, obtained by substituting $r=r_w$ in equation (6) is

$$H/H_0 = F(\beta,\alpha),$$

where

$$F(\beta,\alpha) = (8\alpha/\pi^2)\int_0^{\infty} (\exp(-\beta u^2/\alpha)/u\Delta(u))\, du$$

and where α, β, $\Delta(u)$ are as defined previously.

Comments: Figure 9.1 is a cross section showing geometric configuration along the well bore. The volume of water injected into or discharged from the well is $\pi r_c^2 H_0$. The water-level data in the injected well, expressed as a fraction of H_0, is plotted versus time on semilogarithmic graph paper. This plot is superimposed on figure 9.2, keeping the baselines the same and sliding horizontally until a match or interpolated fit is made. A match point for β, t, and α is picked from the two graphs. Transmissivity is calculated from $T=\beta r_c^2/t$ and storage coefficient from $S = \alpha r_c^2/r_w^2$. As pointed out by Cooper, Bredehoeft, and Papadopulos (1967, p. 267), the determination of S by this method has questionable reliability because of the similar shape of the curves, whereas the determination of T is not as sensitive to choosing the correct curve. Figure 9.2 on plate 1 is plotted from data in table 9.1, which contains original material from two sources (Cooper and others, 1967; and Papadopulos and others, 1973).

Table 9.2 is a listing of a FORTRAN program by S. S. Papadopulos that evaluates $F(\beta,\alpha)$. Input to the program consists of cards coded in a specific format (readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual). Input consists of two or more cards, each containing a single value of

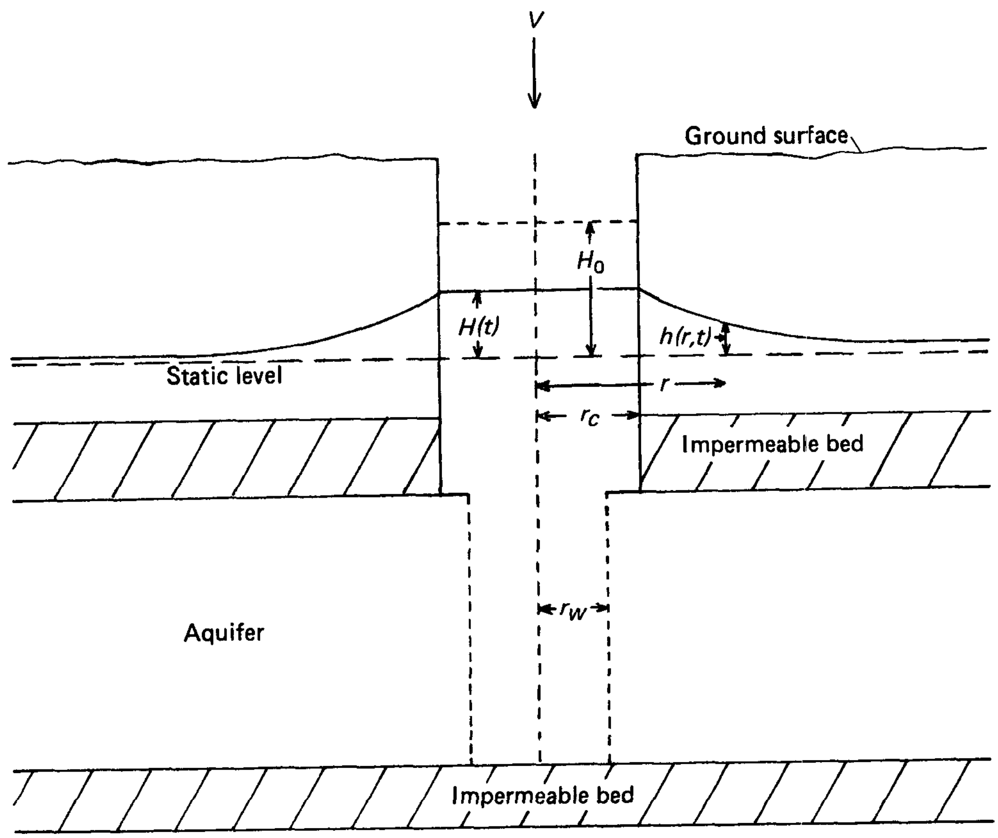

FIGURE 9.1.—Cross section through a well in which a slug of water is suddenly injected.

α coded in format F16.5. The first $\alpha \leqslant 0$ will signal program termination. Output from the program is shown in figure 9.3.

Solution 10: Constant discharge from a fully penetrating well in an aquifer that is anisotropic in the horizontal plane

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is anisotropic in the horizontal plane.
4. Aquifer is not leaky.
5. The transmissivity of the aquifer, T, is a two-dimensional symmetric tensor.

Differential equation:

$$T_{xx}\,\partial^2 s/\partial x^2 + 2T_{xy}\partial^2 s/\partial x\partial y + T_{yy}\partial^2 s/\partial y^2 + Q\,\delta(x)\delta(y) = S\,\partial s/\partial t.$$

This differential equation describes nonsteady flow in a homogeneous anisotropic aquifer with a constantly discharging well at $x=y=0$. The Dirac delta function is represented as $\delta(z)$ and has the following properties: $\delta(z)=0$ if $z\neq 0$ and $\int_{-\infty}^{\infty} \delta(z)dz=1$.

Boundary and initial conditions:

$$s(x,y,0)=0 \qquad (1)$$

$$s(\pm\infty,y,t)=0 \qquad (2)$$

$$s(x,\pm\infty,t)=0 \qquad (3)$$

TABLE 9.1.—*Values of* H/H_0

From Cooper, Bredehoeft, and Papadopulos, 1967						
Tt/r_c^2	α	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
	1.00	0.9771	0.9920	0.9969	0.9985	0.9992
10^{-3}	2.15	.9658	.9876	.9949	.9974	.9985
	4.64	.9490	.9807	.9914	.9954	.9970
	1.00	.9238	.9693	.9853	.9915	.9942
10^{-2}	2.15	.8860	.9505	.9744	.9841	.9883
	4.64	.8293	.9187	.9545	.9701	.9781
	1.00	.7460	.8655	.9183	.9434	.9572
10^{-1}	2.15	.6289	.7782	.8538	.8935	.9167
	4.64	.4782	.6436	.7436	.8031	.8410
	1.00	.3117	.4598	.5729	.6520	.7080
10^{0}	2.15	.1665	.2597	.3543	.4364	.5038
	4.64	.07415	.1086	.1554	.2082	.2620
	7.00	.04625	.06204	.08519	.1161	.1521
	1.00	.03065	.03780	.04821	.06355	.08378
	1.40	.02092	.02414	.02844	.03492	.04426
10^{1}	2.15	.01297	.01414	.01545	.01723	.01999
	3.00	.009070	.009615	.01016	.01083	.01169
	4.64	.005711	.004919	.006111	.006319	.006554
	7.00	.003722	.003809	.003884	.003962	.004046
	1.00	.002577	.002618	.002653	.002688	.002725
10^{2}	2.15	.001179	.001187	.001194	.001201	.001208

From Papadopulos, Bredehoeft, and Cooper, 1973						
Tt/r_c^2	α	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
	1	0.9994	0.9996	0.9996	0.9997	0.9997
	2	.9989	.9992	.9993	.9994	.9995
10^{-3}	4	.9980	.9985	.9987	.9989	.9991
	6	.9972	.9978	.9982	.9984	.9986
	8	.9964	.9971	.9976	.9980	.9982
	1	.9956	.9965	.9971	.9975	.9978
	2	.9919	.9934	.9944	.9952	.9958
10^{-2}	4	.9848	.9875	.9894	.9908	.9919
	6	.9782	.9819	.9846	.9866	.9881
	8	.9718	.9765	.9799	.9824	.9844
	1	.9655	.9712	.9753	.9784	.9807
	2	.9361	.9459	.9532	.9587	.9631
10^{-1}	4	.8828	.8995	.9122	.9220	.9298
	6	.8345	.8569	.8741	.8875	.8984
	8	.7901	.8173	.8383	.8550	.8686
	1	.7489	.7801	.8045	.8240	.8401
	2	.5800	.6235	.6591	.6889	.7139
	3	.4554	.5033	.5442	.5792	.6096
	4	.3613	.4093	.4517	.4891	.5222
10^{0}	5	.2893	.3351	.3768	.4146	.4487
	6	.2337	.2759	.3157	.3525	.3865
	7	.1903	.2285	.2655	.3007	.3337
	8	.1562	.1903	.2243	.2573	.2888
	9	.1292	.1594	.1902	.2208	.2505
	1	.1078	.1343	.1620	.1900	.2178
	2	.02720	.03343	.04129	.05071	.06149
	3	.01286	.01448	.01667	.01956	.02320
10^{1}	4	.008337	.008898	.009637	.01062	.01190
	5	.006209	.006470	.006789	.007192	.007709
	6	.004961	.005111	.005283	.005487	.005735
	8	.003547	.003617	.003691	.003773	.003863
	1	.002763	.002803	.002845	.002890	.002938
10^{2}	2	.001313	.001322	.001330	.001339	.001348

F(BETA,ALPHA) FOR ALPHA= 1.00D-01

BETA	H/HO
1.000D-03	0.9769
2.000D-03	0.9670
4.000D-03	0.9528
6.000D-03	0.9417
8.000D-03	0.9322
1.000D-02	0.9238
2.000D-02	0.8904
4.000D-02	0.8421
6.000D-02	0.8048
8.000D-02	0.7734
1.000D-01	0.7459
2.000D-01	0.6418
4.000D-01	0.5095
6.000D-01	0.4227
8.000D-01	0.3598
1.000D 00	0.3117
2.000D 00	0.1786
3.000D 00	0.1196
4.000D 00	0.0876
5.000D 00	0.0681
6.000D 00	0.0553
7.000D 00	0.0463
8.000D 00	0.0396
9.000D 00	0.0346
1.000D 01	0.0306
2.000D 01	0.0141
3.000D 01	0.0091
4.000D 01	0.0067
5.000D 01	0.0053
6.000D 01	0.0044
7.000D 01	0.0037
8.000D 01	0.0032
9.000D 01	0.0029
1.000D 02	0.0026
2.000D 02	0.0013
4.000D 02	0.0006
6.000D 02	0.0004
8.000D 02	0.0003
1.000D 03	0.0003

FIGURE 9.3.—Example of output from program to compute change in water level due to sudden injection of a slug of water into a well.

Equation 1 states that, initially, drawdown is zero. Equations 2 and 3 state that the drawdown approaches zero as distance from the discharging well approaches infinity, a condition which will be approximated if boundaries of the aquifer are sufficiently distant from the discharging well.

Solution (Papadopulos, 1965, p. 23):

$$s = (Q/4\pi\sqrt{T_{xx}T_{yy}-T_{xy}^2})\, W(u_{xy}), \qquad (4)$$

where

$$W(u) = \int_u^{\infty} (e^{-v}/v)\, dv$$

and

$$u_{xy} = (S/4t)(T_{xx}y^2 + T_{yy}x^2 - 2T_{xy}xy)/(T_{xx}T_{yy} - T_{xy}^2). \qquad (5)$$

If the coordinate axes x and y are the same as the principal axes ϵ and η (fig. 10.1) of the transmissivity tensor, the preceding equation for drawdown becomes

$$s = (Q/4\pi\sqrt{T_{\epsilon\epsilon}\, T_{\eta\eta}})\, W(u_{\epsilon\eta}),$$

where

$$u_{\epsilon\eta} = (S/4t)(T_{\epsilon\epsilon}\, n^2 + T_{\eta\eta}\, \epsilon^2)/T_{\epsilon\epsilon}\, T_{\eta\eta} .$$

Comments: The method of type-curve solution as outlined by Papadopulos (1965, p. 26) requires observation of drawdown in at least three observation wells. First, choose a convenient rectangular coordinate system with the pumped well at the origin. Then, plot the observed drawdown versus t on logarithmic paper. Match these plots to the $W(u)$ type curve given in solution 1. Choose a match point of (t,s) and $(1/u_{xy}, W(u_{xy}))$ for each well and compute $T_{xx}T_{yy}-T_{xy}^2 = (QW(u_{xy})/4\pi s)^2$ for each well. Match points for all observation wells should yield approximately the same value of $(T_{xx}T_{yy}-T_{xy}^2)$. Usually they will not and judgment must be used to obtain an "average" value. Substituting this value and the three values of (x,y) in equation 5 gives three equations in three unknowns ST_{xx}, ST_{yy}, and ST_{xy}. These equations are of the form

$$y^2(ST_{xx}) + x^2(ST_{yy}) - 2xy(ST_{xy}) = 4tu_{xy}(T_{xx}T_{yy} - T_{xy}^2).$$

Solve these three equations to determine T_{xx}, T_{xy}, and T_{yy} in terms of S, and S may be determined from

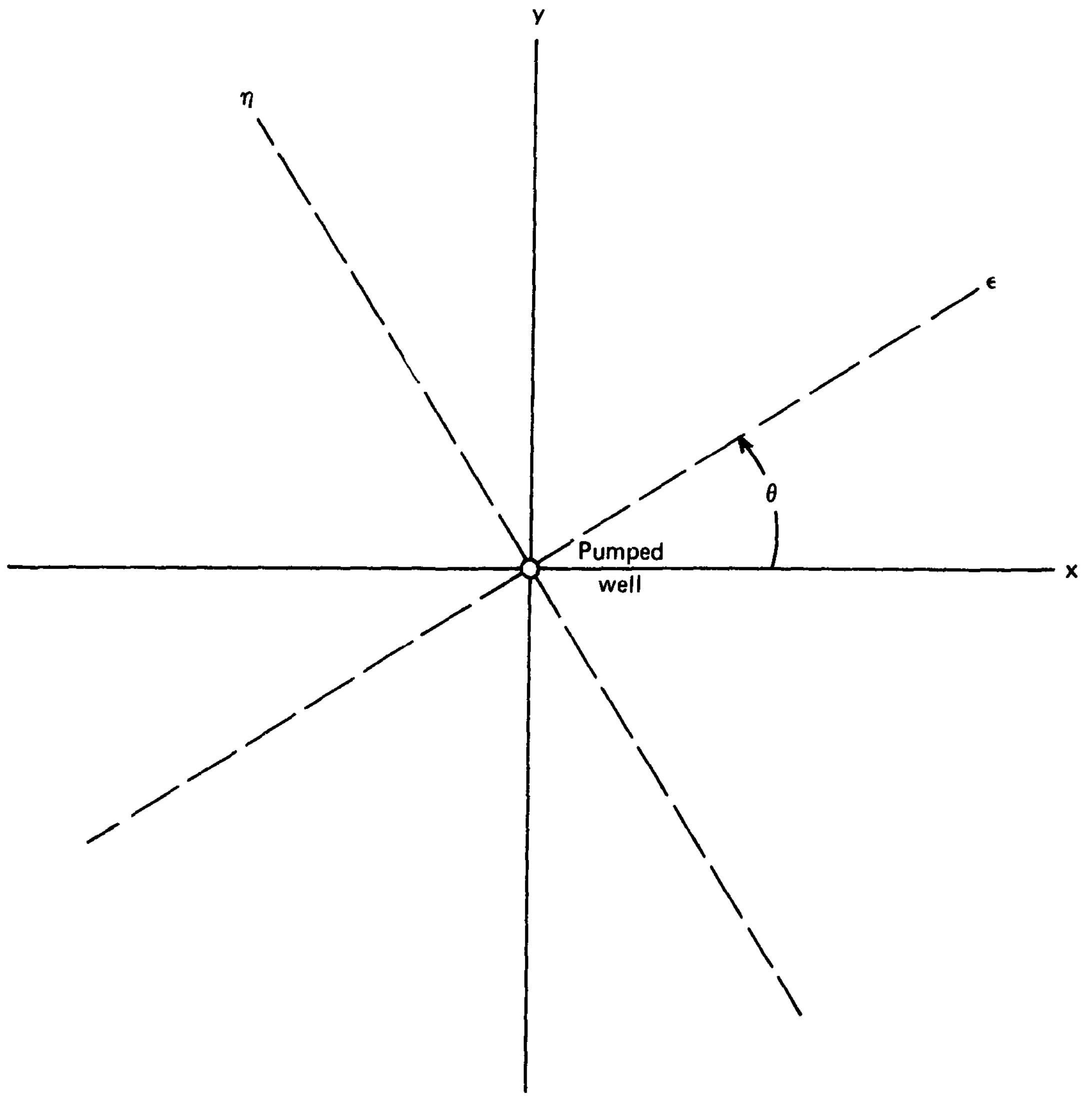

FIGURE 10.1.—Plan view showing coordinate axes.

$S = \sqrt{(ST_{xx}ST_{yy} - (ST_{xy})^2)/(T_{xx}T_{yy} - T_{xy}^2)}.$

Then, compute T_{xx}, T_{yy}, and T_{xy} from ST_{xx}, ST_{yy}, and ST_{xy}. $T_{\epsilon\epsilon}$, $T_{\eta\eta}$, and Θ (the angle between the x and the ϵ axis) may be calculated from the relations (Papadopulos, 1965, p. 28)

$$T_{\epsilon\epsilon} = 1/2(T_{xx} + T_{yy} + ((T_{xx} - T_{yy})^2 + 4T_{xy}^2)^{1/2})$$

$$T_{\eta\eta} = 1/2(T_{xx} + T_{yy} - ((T_{xx} - T_{yy})^2 + 4T_{xy}^2)^{1/2})$$

$$\Theta = \arctan((T_{\epsilon\epsilon} - T_{xx})/T_{xy}).$$

Solution 11: Variable discharge from a fully penetrating well in a leaky aquifer

Assumptions:

1. Well discharge changes as a specified function of time.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').

4. Confining bed is overlain, or underlain, by an infinite constanthead plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption will be approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{sK'}{Tb'} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,0)=0 \qquad (1)$$

$$s(\infty,t)=0 \qquad (2)$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q(t)}{2\pi T},\ t \geqslant 0 \qquad (3)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is zero at large distances from the pumped well. Equation 3 states that near the pumped well the radial flow is equal to the discharge of the pumped well, which is a function of time.

Solution:

Solutions for certain discharge functions have been published by Abu-Zied and Scott (1963), and Werner (1946) for a nonleaky aquifer, and by Hantush (1964a) for both leaky and nonleaky aquifers. For arbitrary discharge functions for leaky aquifers, a solution using the convolution integral has been presented by Moench (1971, eq. 3):

$$s = (1/4\pi T)\int_0^t (Q(t')/(t-t')) \cdot \exp\left(-A/(t-t') - (t-t')K'/Sb'\right)dt', \qquad (4)$$

where $Q(t)$ is the discharge function of time and $A = r^2S/4T$. A numerical integration scheme is generally necessary to evaluate the above equation.

For type curves, a more useful form of equation 4 is

$$s = (Q_r/4\pi T)\int_0^t \left[Q(t')/Q_r\ (t-t')\right] \cdot \exp\left[-A/(t-t')-(t-t')K'/Sb'\right]dt', \qquad (5)$$

or

$$s = (Q_r/4\pi T)\ SO(t), \qquad (6)$$

where $SO(t)$, read "system output function," represents the integral expression in equation 5, and Q_r is an arbitrary discharge that eliminates dimension from the integral expression. For example, Q_r could be the initial, final, or average discharge, according to the needs of the user.

Comments: Figure 11.1 is a cross section through the discharging well. This situation is the same as for solution 4, except for the varying discharge of the well. The effect of finite well radius (r_w) was investigated by Hantush (1964b, p. 4224), who concluded that for $t>25r_w^2S/T$ and $r_w/\sqrt{Tb'/K'} < 0.1$ the drawdown could be represented closely by the convolution integral.

Figure 11.2 on plate 1 shows a selected set of type curves for linear change in discharge in a nonleaky aquifer. The solution for this type of discharge function has been presented by Werner (1946, p. 706). The discharge function for figure 12.2 is $Q(t)=Q_0(1+ct)$, and the resulting drawdown is

$$s = (Q_0/4\pi T)W(u)\left\{1+ct\left[u+1-e^{-u}/W(u)\right]\right\},$$

where $W(u)$ is the well function of Theis. Substituting A/u for t in the above expression gives

$$s = (Q_0/4\pi T)\ W(u) \cdot \left(1+cA\left\{1+(1/u)\left[1-e^{-u}/W(u)\right]\right\}\right),$$

or

$$s = (Q_0/4\pi T)\ SO(t),$$

where $SO(t)$ represents

$$W(u)\left(1+cA\left\{1+(1/u)\left[1-e^{-u}/W(u)\right]\right\}\right).$$

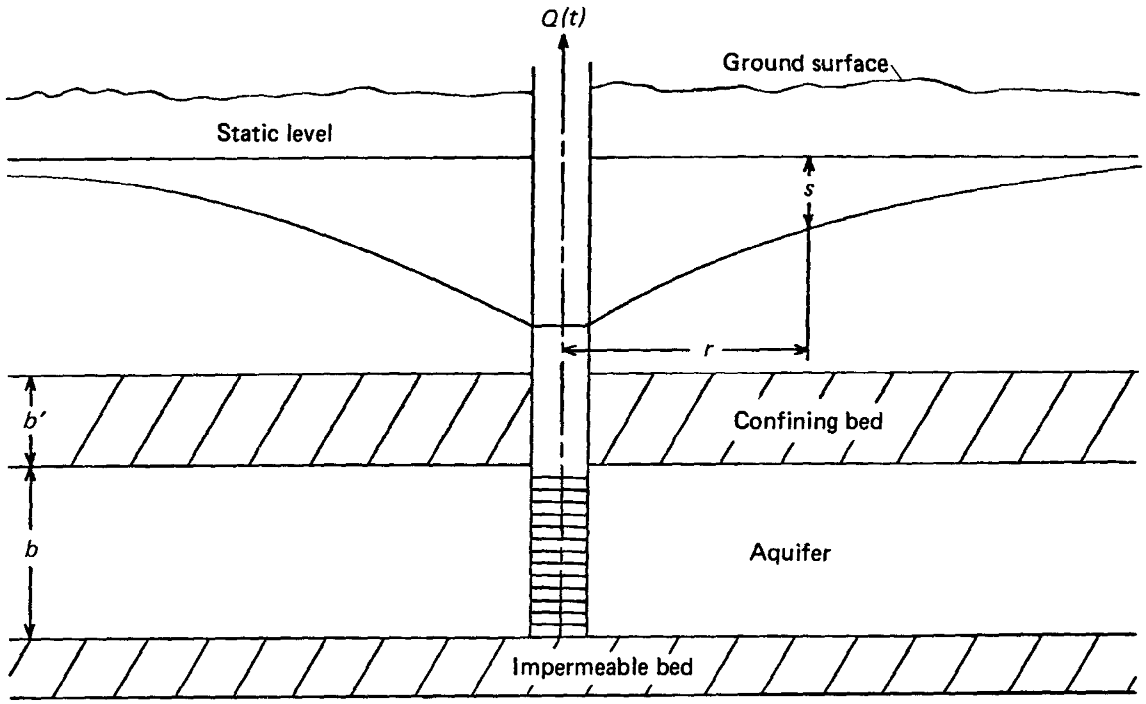

FIGURE 11.1.—Cross section through a well with variable discharge.

This substitution permits the plotting of a family of type curves, each curve specified by a value of cA.

Table 11.1 is the listing of a FORTRAN program designed to evaluate the above convolution integral for five different discharge functions. Three of these discharge functions are those devised by Hantush (1964a, p. 343, 344), who presented solutions for drawdown resulting from these functions. These three discharge functions are:

(a) $Q(t)=Q_s\left[1+\delta \exp\left(-t/t^*\right)\right]$,

(b) $Q(t)=Q_s\left[1+\delta/(1+t/t^*)\right]$,

and (c) $Q(t)=Q_s\left[1+\delta/\sqrt{1+t/t^*}\right]$,

where Q_s is the ultimate steady discharge and δ and t^* are parameters defining a particular function. The first discharge function, for an exponentially decreasing discharge (case "a" of Hantush, 1964a) is virtually the same as the discharge function of Abu-Zied and Scott (1963). Besides the three functions of Hantush, the program also includes discharge as a fifth-degree polynomial of time, $Q(t)=\sum_{i=0}^{5} a_i t^i$, where the a_i are the coefficients of the polynomial, and as a piecewise linear function of time with eight segments,

$$Q(t)=a_j+b_j(t-t_{j-i})$$

for

$$t_{j-1}<t\leq t_j,\ j=1,\ 2,\ \ldots,\ 8,$$

where a_j and b_j are parameters defining the j^{th} line segment. The program uses a different, but equivalent to equation 4, expression for the convolution integral

$$s = (1/4\pi T)\int_0^t (Q(t-t')/t') \cdot \exp\left(-A/t'-t'K'/Sb'\right)\,dt'.$$

The program uses a sum to approximate the convolution integral. It chooses a starting value of t' that satisfies $r^2S/4Tt' + K't'/Sb' = 100$. If such a value of t' does not exist, that is, $(r^2S/4T)\,(K'/Sb')>2500$, then a value of zero is assigned for the integral value. The ending point of the interval is picked as 10 times the starting point. The integral over this interval is approximated by a trapezoidal sum using 500 subdivisions of the interval. A new interval is then constructed using the previous end point as a new starting point and a new ending point equal to 10 times the new starting point. This new interval is again evaluated by a trapezoidal sum of 500 segments. This summation procedure over intervals that are successively an order of magnitude larger continues until either $t'=t$ or $(r^2S/4Tt') + (K't/Sb') > 101$. Input to this program consists of cards coded in specific formats. Readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual. Input consists of one or more groups of data, each group consisting of the following. First, one card containing the beginning time of the period of analysis in columns 1–10, coded in format E10.3; the ending time coded in columns 1311–20, in format E10.3; and a discharge index (a number from 1 through 5) coded in column 25, in format I1; and a reference discharge, QR, coded in columns 31–40, in format E10.3. The discharge index, IQ, selects a discharge function, $Q(t)$, in the following manner. If $IQ=1$, the discharge function is exponentially decreasing,

$$Q(t)=Q_s\left[1+\delta\exp(-t/t^*)\right].$$

This is case (a) of Hantush (1964a, p. 343). If $IQ=2$, the discharge function is hyperbolically decreasing,

$$Q(t)=Q_s\left[1+\delta/(1+t/t^*)\right].$$

This is case (b) of Hantush (1964a, p. 344). If $IQ=3$, the discharge function is the same as case (c) of Hantush (1964a, p. 344),

$$Q(t)=Q_s\left[1+\delta/\sqrt{1+t/t^*}\right].$$

If $IQ=4$, the discharge function is a fifth-degree polynomial of time,

$$Q(t) = \sum_{i=0}^{5} a_i t^i.$$

If $IQ=5$, the discharge function is a piecewise-linear function of time with eight or less segments,

$$Q(t)=a_j+b_j(t-t_{j-1})$$

for $t_{j-1}<t\leq t_j,\ j=1,2,\ldots,8.$

The reference discharge, QR, is used to determine the form of the output from the program: If QR is coded as zero (or blank), the output shows t, s (as defined by eq. 4), and $Q(t)$. If a value greater than zero is coded for QR, the output shows $1/u$, $SO(t)$ (as defined by eq. 6), and $Q(t)/QR$.

Second, there are one or more cards containing parameters of the discharge function. If $IQ=1$, 2, or 3, then it consists of one card containing: QST, the ultimate steady discharge, coded in columns 1–10, in format E10.3; DELTA, a rate parameter, coded in columns 11–20, in format E10.3; TSTAR, a time parameter, coded in columns 21–30, in format E10.3. If $IQ=4$, it is one card containing the six polynomial coefficients. They are coded in the order $a_0, a_1, \ldots, a_5$, in columns 1–10; 11–20, ..., 51–60 all in format E10.3. If $IQ=5$, then the program requires four cards, each card containing t_j, a_j, b_j, t_{j+1}, a_{j+1}, b_{j+1}; the four cards representing $j=1, 3, 5, 7$. The last part of each set of data consists of two or more cards containing coded values for: distance from pumped well, in columns 1–10; storage coefficient, in columns 11–20; transmissivity, in columns 21–30; and ratio of hydraulic conductivity to thickness for the confining bed, in columns 31–40, all in format E10.3. A blank card is used to signal the end of each set of data. Output from this program is shown in figure 11.3.

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