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Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B3

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

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Book 3
APPLICATIONS OF HYDRAULICS

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an example of which is shown in figure 2.4. Subroutines DQL12, BESK, and EXPI are from the IBM Scientific Subroutine Package and a discussion of them is in the IBM SSP manual.

Solution 3: Constant drawdown in a well in a nonleaky aquifer

Assumptions:

1. Water level in well is changed instantaneously by s_w at $t = 0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic confined aquifer.

Boundary and initial conditions:

$$s(r,0) = 0,\ r \geq r_w \qquad (1)$$

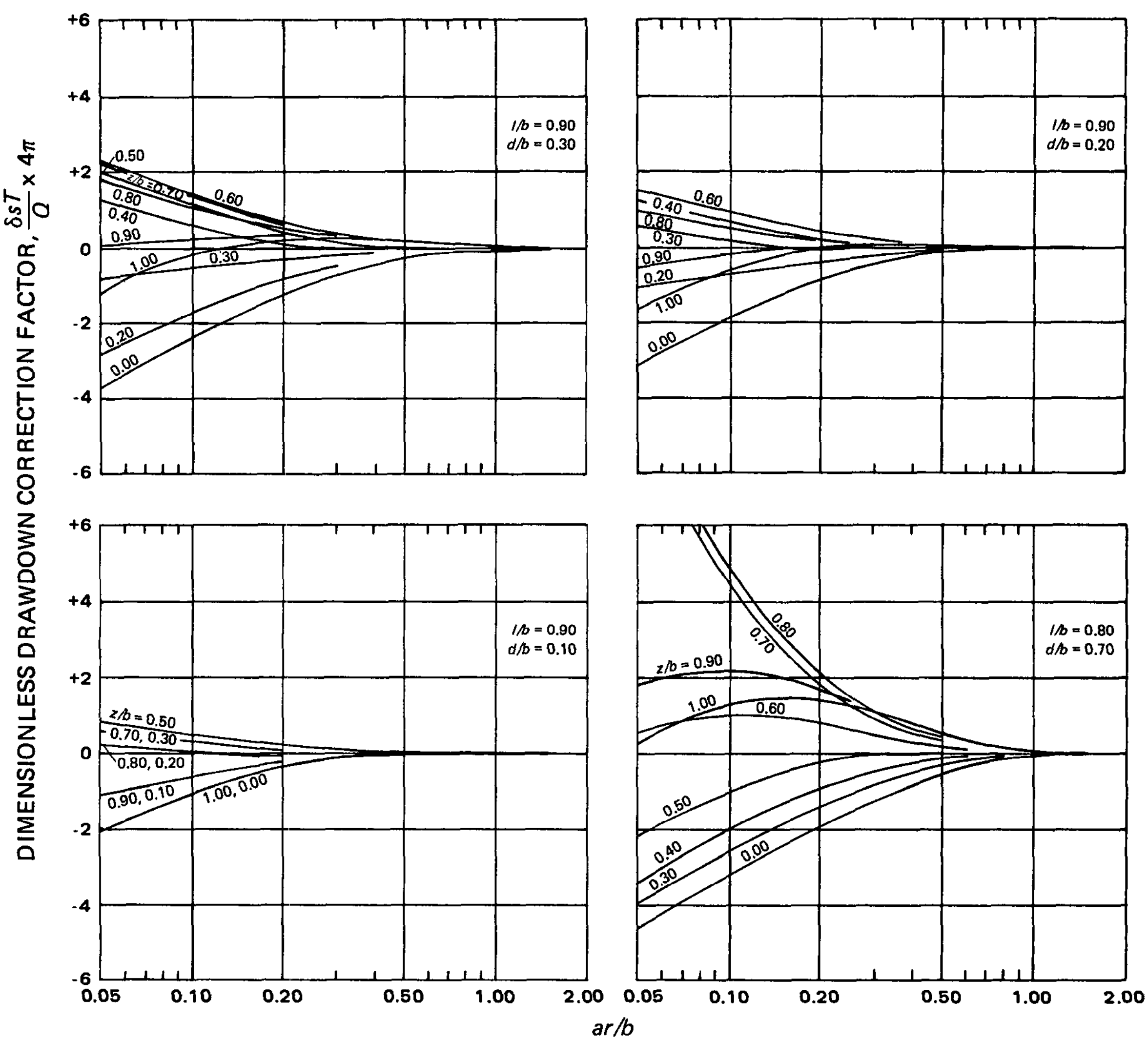

FIGURE 2.3.—Continued.

$$s(r_w,t) = \begin{cases} 0, & t < 0 \\ s_w = \text{constant}, & t \geqslant 0 \end{cases} \tag{2}$$

$$s(\infty,t) = 0,\ t \geqslant 0 \tag{3}$$

Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that, as the well is approached, drawdown in the aquifer approaches the constant drawdown in the well, implying no entrance loss to the well. Equation 3 states that the drawdown approaches zero as the distance from the well approaches infinity.

Solutions:

I. For the well discharge (Jacob and Lohman, 1952, p. 560):

$$Q = 2\pi T\, s_w\, G(\alpha),$$

where

$$G(\alpha) = \frac{4\alpha}{\pi}\int_0^\infty xe^{-\alpha x^2}\left\{\frac{\pi}{2} + \tan^{-1}\left[\frac{Y_0(x)}{J_0(x)}\right]\right\}dx$$

and $$\alpha = \frac{Tt}{Sr_w^2}.$$

II. For the drawdown in water level (Hantush, 1964a, p. 343):

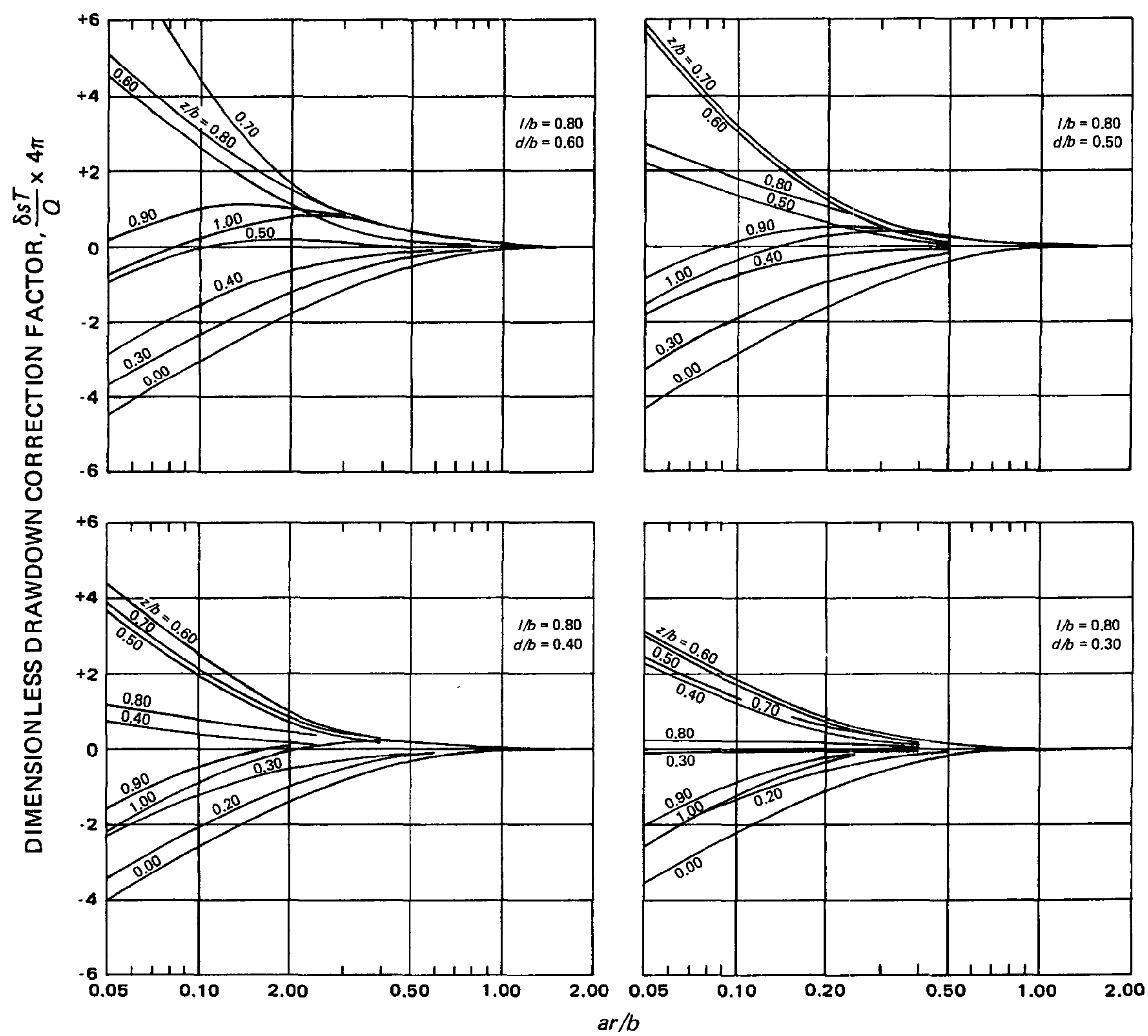

FIGURE 2.3.—Continued.

$$s = s_w\, A(\tau, \rho),$$

where $A(\tau,\rho) = 1$

$$-\frac{2}{\pi}\int_0^{\infty}\frac{J_0(u)\,Y_0(\rho u) - Y_0(u)J_0(\rho u)}{J_0^2(u) + Y_0^2(u)}\exp(-\tau u^2)\frac{du}{u},$$

and $$\tau = \alpha = \frac{Tt}{Sr_w^2},$$

$$\rho = \frac{r}{r_w}.$$

Comments:

Boundary condition 2 requires a constant drawdown in the discharging well, a condition most commonly fulfilled by a flowing well, although figure 3.1 shows the water level to be below land surface.

Figure 3.2 on plate 1 is a plot from Lohman (1972, p. 24) of dimensionless discharge ($G(\alpha)$) versus dimensionless time (α). Additional values in the range α greater than 1×10^{12} were calculated from $G(\alpha)\simeq 2/\log(2.2458\alpha)$ (Hantush, 1964a, p. 312). Function values for $G(\alpha)$ are given in table 3.1. The data curve consists of measured well discharge versus time. After the data and type curves are matched, transmissivity can be calculated from $T = Q/2\pi s_w G(\alpha)$, and the storage coefficient can be

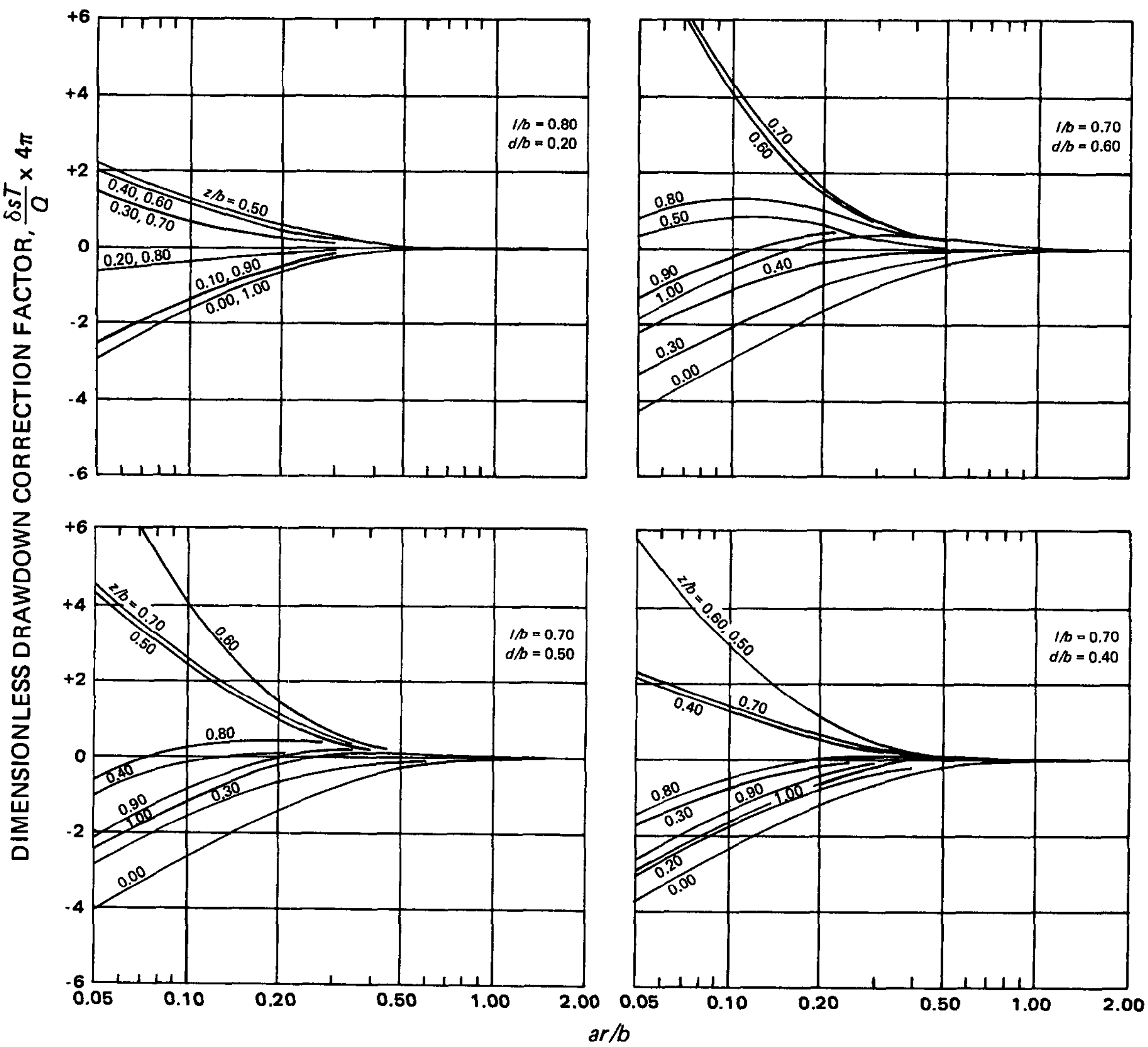

FIGURE 2.3.—Continued.

calculated from $S = Tt/\alpha r_w^2$, where $(\alpha, G(\alpha))$ and (t,Q) are matching points on the type curve and data curve, respectively.

Similarly, data curves of drawdown versus time may be matched to figure 3.3 on plate 1; this is a plot of dimensionless drawdown $(A(\tau,\rho)=s/s_w)$ versus dimensionless time $(\tau/\rho^2 = Tt/Sr^2)$. After the data and type curves are matched, the hydraulic diffusivity of the aquifer can be calculated from the equality $T/S = (\tau/\rho^2)(r^2/t)$. Usually s_w is known, and some of the uncertainty of curve matching can be eliminated by plotting s/s_w versus t because only horizontal translation is then required. If r_w is also known, the particular curve to be matched can be determined from the relation $\rho = r/r_w$. Generally, however, the effective radius, r_w, differs from the actual radius and is not known. The effective radius can often be estimated from a knowledge of the construction of the well and the water-bearing material, or it can be determined from step-drawdown tests (Rorabaugh, 1953). Figure 3.3 was plotted from table 3.2. For $\tau \leq 1 \times 10^3$, the data are from Hantush (1964a, p. 310). For $\tau > 1 \times 10^3$, values of drawdown in a leaky aquifer, as $r_w/B \rightarrow 0$, were used. (See solution 7.) Where 0.000 occurs in table 3.2, $A(\tau,\rho)$ is less than 0.0005.

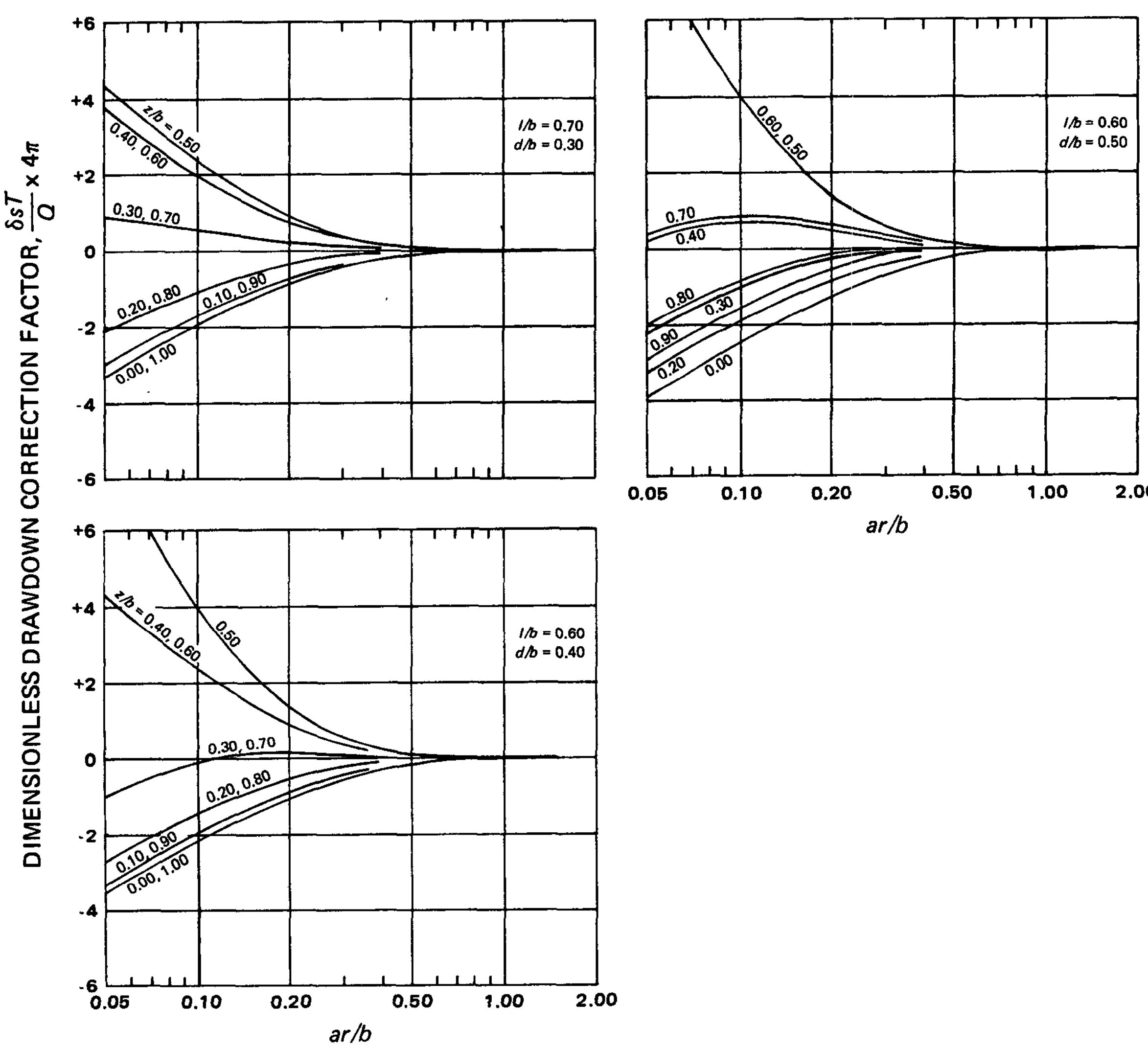

FIGURE 2.3.—Continued.

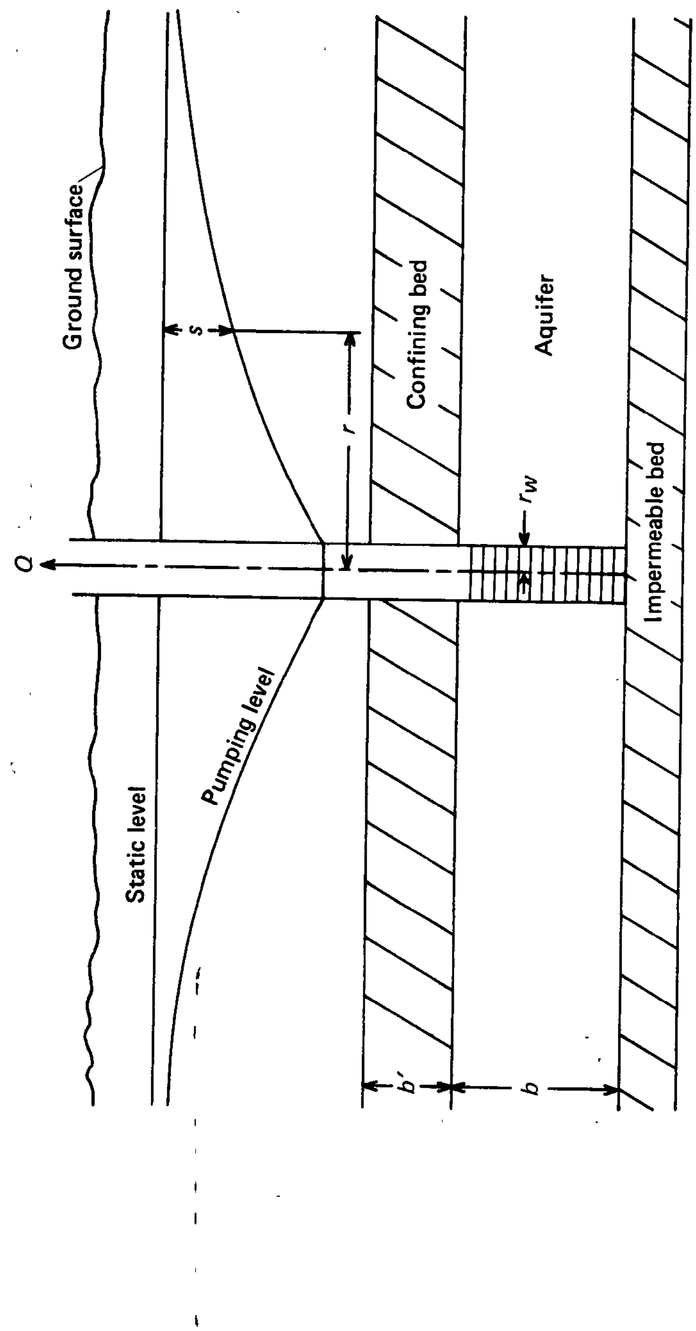

FIGURE 2.4.—Example of output from program for partial penetration in a nonleaky artesian aquifer.

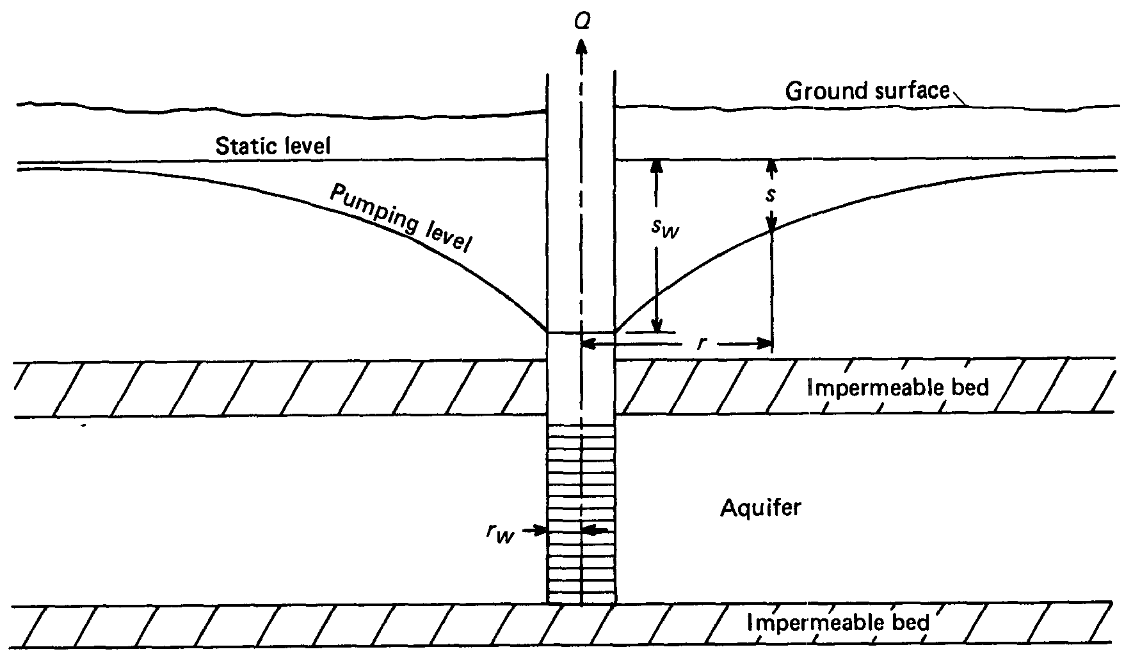

FIGURE 3.1.—Cross section through a well with constant drawdown in a nonleaky aquifer.

Solution 4: Constant discharge from a fully penetrating well in a leaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{sK'}{Tb'} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

$$\tag{1}$$

$$s(\infty,t)=0,\ t\geqslant 0 \tag{2}$$

$$Q=\begin{cases}0, & t<0\\ \text{constant}>0, & t\geqslant 0\end{cases} \tag{3}$$

$$\lim_{r\to 0} r\,\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \tag{4}$$

Equation 1 states that the initial drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equation 3 states that the discharge from the well is constant and begins at $t=0$. Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.

TABLE 3.1.—*Values of* G(α)

[Modified from Lohman (1972, p 24)]

α	$\alpha \times 10^{-4}$	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}
1	56 9	18 34	6.13	2.249	0 985	0 534	0 346	0 251	0 1964	0 1608
2	40 4	13.11	4 47	1.716	803	461	311	.232	1841	1524
3	33 1	10 79	3 74	1 477	719	427	.294	.222	1777	.1479
4	28 7	9 41	3 30	1 333	.667	405	283	.215	.1733	.1449
5	25 7	8 47	3 00	1 234	630	389	274	.210	.1701	.1426
6	23 5	7 77	2.78	1 160	.602	377	268	206	1675	1408
7	21.8	7 23	2 60	1 103	580	367	.263	203	.1654	1393
8	20.4	6.79	2 46	1 057	562	359	.258	200	.1636	1380
9	19.3	6 43	2 35	1 018	547	352	254	198	1621	.1369
α	$\alpha \times 10^{6}$	10^{7}	10^{8}	10^{9}	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}
1	0 1360	0 1177	0.1037	0 0927	0 0838	0 0764	0 0704	0 0651	0 0605	0 0566
2	1299	.1131	1002	.0899	.0814	.0744	0686	.0636	0593	.0555
3	1266	1106	.0982	0883	0801	0733	.0677	0628	0586	.0549
4	1244	.1089	0968	0872	.0792	0726	.0671	.0622	0581	
5	.1227	1076	0958	0864	0785	.0720	0666	.0618	0577	
6	1213	1066	0950	.0857	0779	.0716	.0662	.0615	0574	
7	.1202	.1057	0943	0851	.0774	.0712	.0658	0612	.0572	
8	1192	.1049	0937	0846	0770	.0709	.0655	.0609	.0569	
9	1184	1043	0932	.0842	0767	0706	0653	.0607	0567	

TABLE 3.2.—*Values of* $A(\tau,\rho)$

[Values of $A(\tau,\rho)$ for $\tau \leq 10^3$ modified from Hantush (1964a, p 310)]

τ		ρ							
		5	10	20	50	100	200	500	1000
1	$\times 1$	0.002							
2		.022							
3		.049							
4		.076	0.000						
5		.101	.002						
7		.142	.006						
1	$\times 10$	.188	.016	0.000					
2		.277	.057	.001					
3		.325	.094	.004					
4		.358	.123	.009					
5		.381	.146	.016					
7		.414	.184	.031					
1	$\times 10^2$	.446	.222	.053	0.000				
1.5		.479	.264	.085	.001				
2		.500	.291	.110	.003				
3		.528	.328	.146	.009				
5		.559	.372	.194	.026	0.000			
7		.578	.397	.223	.044	.001			
1	$\times 10^3$	.596	.422	.254	.066	.004			
1.5		.615	.450	.287	.094	.012			
2		.627	.467	.309	.116	.021	0.000		
3		.644	.490	.338	.147	.039	.001		
5		.662	.517	.372	.186	.068	.006		
7		.673	.533	.392	.211	.089	.014		
1	$\times 10^4$	.685	.549	.413	.237	.114	.025		
1.5		.696	.566	.435	.264	.142	.043	0.000	
2		.704	.577	.450	.283	.161	.058	.001	
3		.715	.592	.469	.308	.188	.081	.005	
5		.727	.609	.492	.337	.221	.113	.014	0.000
7		.734	.620	.506	.355	.242	.134	.025	.001
1	$\times 10^5$	.742	.631	.520	.373	.263	.156	.039	.002
1.5		.750	.642	.532	.392	.285	.180	.058	.007
2		.755	.650	.544	.405	.300	.197	.072	.013
3		.762	.660	.558	.423	.321	.220	.094	.024
5		.771	.672	.574	.443	.345	.247	.122	.044
7		.776	.680	.584	.456	.360	.264	.141	.059
1	$\times 10^6$	.782	.688	.594	.470	.376	.282	.160	.076
1.5		.788	.696	.604	.484	.392	.301	.181	.096
2		.792	.702	.612	.493	.403	.314	.196	.111
3		.797	.709	.622	.506	.418	.331	.216	.132
5		.803	.718	.633	.521	.436	.352	.240	.157
7		.807	.724	.641	.531	.448	.365	.255	.173
1	$\times 10^7$	.811	.730	.648	.541	.459	.378	.270	.190
1.5		.815	.736	.656	.551	.472	.392	.287	.208
2		.818	.740	.662	.558	.480	.402	.299	.221
3		.822	.746	.669	.568	.492	.415	.314	.238
5		.827	.753	.678	.580	.506	.431	.333	.258
7		.830	.757	.684	.587	.514	.441	.344	.271
1	$\times 10^8$	.833	.762	.690	.595	.523	.452	.357	.285
1.5		.837	.766	.696	.603	.533	.463	.370	.300
2		.839	.770	.701	.609	.540	.470	.379	.310
3		.842	.774	.706	.617	.549	.481	.391	.323
5		.846	.780	.714	.626	.560	.494	.406	.340
7		.849	.783	.718	.632	.567	.502	.415	.350
1	$\times 10^9$	.851	.787	.723	.638	.574	.510	.425	.361
1.5		.854	.791	.728	.645	.582	.519	.435	.372
2		.856	.794	.731	.649	.587	.525	.443	.380
3		.858	.797	.736	.655	.594	.533	.452	.392
5		.861	.802	.742	.663	.603	.544	.464	.405
7		.863	.804	.746	.668	.609	.550	.472	.413
1	$\times 10^{10}$	.865	.807	.749	.673	.615	.557	.480	.422
1.5		.867	.810	.753	.678	.621	.564	.488	.431
2		.869	.813	.756	.682	.625	.569	.494	.438
3		.871	.816	.760	.687	.631	.576	.502	.447
5		.874	.819	.765	.693	.638	.584	.512	.457
7		.875	.821	.768	.696	.643	.589	.518	.464
1	$\times 10^{11}$	.877	.824	.770	.700	.647	.594	.524	.471
1.5									
2									
3									
5									
7									

Solution (Hantush and Jacob, 1955, p. 98):

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-z-\frac{r^2}{4B^2z}}}{z} \qquad (5)$$

where $u = r^2S/4Tt$

$$B = \sqrt{\frac{Tb'}{K'}}. \qquad (6)$$

Comments:

As pointed out by Hantush and Jacob (1954, p. 917), leakage is three-dimensional, but if the difference in hydraulic conductivities of the aquifer and confining bed are sufficiently great, the flow may be assumed to be vertical in the confining bed and radial in the aquifer. This relationship has been quantified by Hantush (1967, p. 587) in the condition $b/B<0.1$. In terms of relative conductivities, this would be $K/K' > 100\ b/b'$. Assumption 5, that there is no change in storage of water in the confining bed, was investigated by Neuman and Witherspoon (1969b, p. 821). They concluded that this assumption would not affect the solution if

$$\beta < 0.01, \quad \text{where } \beta = \frac{r}{4b}\sqrt{\frac{K'S_s'}{KS_s}}.$$

Assumption 4, that there is no drawdown in water level in the source bed lying above the confining bed, was also examined by Neuman and Witherspoon (1969a, p. 810). They indicated that drawdown in the source bed would have negligible effect on drawdown in the pumped aquifer for short times, that is, when $\frac{Tt}{r^2S} < 1.6\frac{\beta^2}{(r/B)^4}$. Also, they indicated (1969a, p. 811) that neglect of drawdown in the source bed is justified if $T_s > 100T$, where T_s represents the transmissivity of the source bed. Figure 4.1, a cross section through the discharging well, shows geometric relationships. Figure 4.2 on plate 1 shows plots of dimensionless drawdown compared to dimensionless time, using the notation of Cooper (1963) from Lohman (1972, pl. 3). Cooper expressed equations 5 and 6 as

$$L(u,v) = \int_u^\infty \frac{e^{-y-\frac{v^2}{y}}}{y}\,dy, \qquad (7)$$

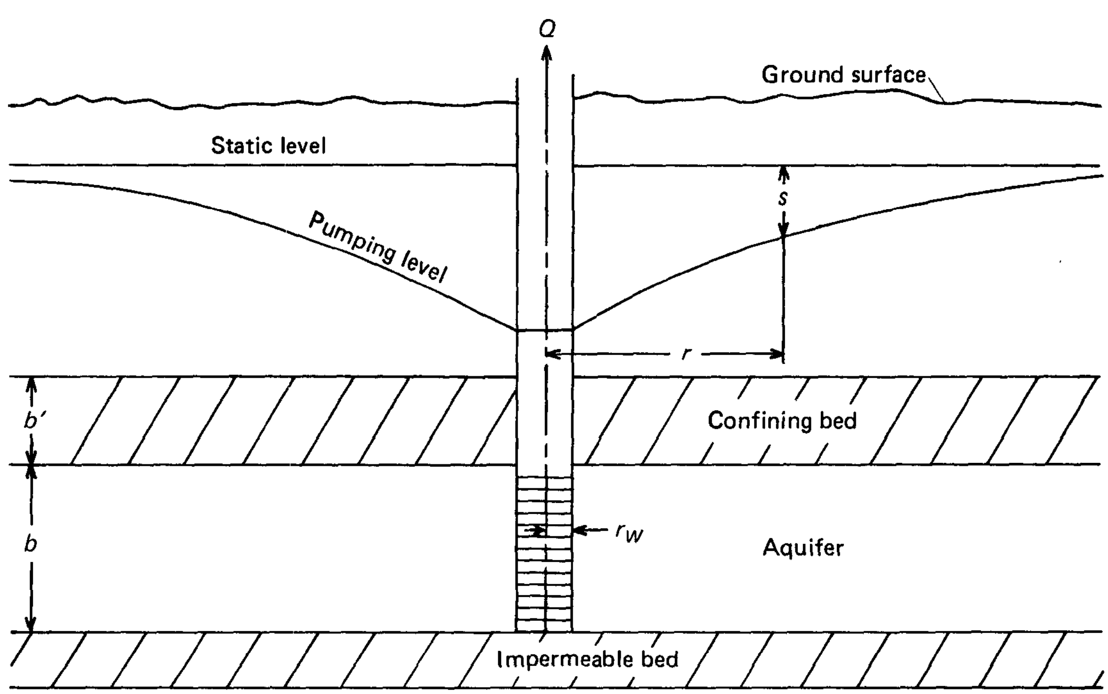

FIGURE 4.1.—Cross section through a discharging well in a leaky aquifer.

with

$$v = \frac{r}{2}\sqrt{\frac{K'}{Tb'}}. \tag{8}$$

Cooper's type curves and equation 5 express the same function with $r/B=2v$. Hantush (1961e) has a tabulation of equation 5, parts of which are included in table 4.1.

The observed data may be plotted in two ways (Cooper, 1963, p. C51). The measured drawdown in any one well is plotted versus t/r^2; the data are then matched to the solid-line type curves of figure 4.2. The data points are alined with the solid-line type curves either on one of them or between two of them. The parameters are then computed from the coordinates of the match points $(t/r^2, s)$ and $(1/u, L(u,v))$, and an interpolated value of v from the equations

$$T = \frac{Q}{4\pi}\frac{L(u,v)}{s}, \tag{9}$$

$$S = 4T\frac{t/r^2}{1/u}, \tag{10}$$

and

$$\frac{K'}{b'} = 4T\frac{v^2}{r^2}.$$

Drawdown measured at the same time but in different observation wells at different distances can be plotted versus t/r^2 and matched to the dashed-line type curves of figure 4.2. The data are matched so as to aline with the dashed-line curves, either on one or between two of them. From the match-point coordinates $(s, t/r^2)$ and $(L(u,v), 1/u)$ and an interpolated value of v^2/u, T and S are computed from equations 9 and 10 and the remaining parameter from

$$K'/b' = S\frac{v^2/u}{t}.$$

The region $v^2/u \geqslant 8$ and $L(u,v) \geqslant 10^{-2}$ corresponds to steady-state conditions.

TABLE 4.1.—*Selected values of* W(u,r/B)

[From Hantush (1961e)]

u	r/B = 0.001	0.003	0.01	0.03	0.1	0.3	1	3
1×10^{-6}	13.0031	11.8153	9.4425	7.2471	4.8541	2.7449	0.8420	0.0695
2	12.4240	11.6716						
3	12.0581	11.5098	9.4425					
5	11.5795	11.2248	9.4413					
7	11.2570	10.9951	9.4361					
1×10^{-5}	10.9109	10.7228	9.4176					
2	10.2301	10.1332	9.2961	7.2471				
3	9.8288	9.7635	9.1499	7.2470				
5	9.3213	9.2818	8.8827	7.2450				
7	8.9863	8.9580	8.6625	7.2371				
1×10^{-4}	8.6308	8.6109	8.3983	7.2122				
2	7.9390	7.9290	7.8192	7.0685				
3	7.5340	7.5274	7.4534	6.9068	4.8541			
5	7.0237	7.0197	6.9750	6.6219	4.8530			
7	6.6876	6.6848	6.6527	6.3923	4.8478			
1×10^{-3}	6.3313	6.3293	6.3069	6.1202	4.8292			
2	5.6393	5.6383	5.6271	5.5314	4.7079	2.7449		
3	5.2348	5.2342	5.2267	5.1627	4.5622	2.7448		
5	4.7260	4.7256	4.7212	4.6829	4.2960	2.7428		
7	4.3916	4.3913	4.3882	4.3609	4.0771	2.7350		
1×10^{-2}	4.0379	4.0377	4.0356	4.0167	3.8150	2.7104		
2	3.3547	3.3546	3.3536	3.3444	3.2442	2.5688		
3	2.9591	2.9590	2.9584	2.9523	2.8873	2.4110	.8420	
5	2.4679	2.4679	2.4675	2.4642	2.4271	2.1371	.8409	
7	2.1508	2.1508	2.1506	2.1483	2.1232	1.9206	.8360	
1×10^{-1}	1.8229	1.8229	1.8227	1.8213	1.8050	1.6704	.8190	
2	1.2226	1.2226	1.2226	1.2220	1.2155	1.1602	.7148	.0695
3	.9057	.9057	.9056	.9053	.9018	.8713	.6010	.0694
5	.5598	.5598	.5598	.5596	.5581	.5453	.4210	.0681
7	.3738	.3738	.3738	.3737	.3729	.3663	.2996	.0639
1×10^{0}	.2194	.2194	.2194	.2193	.2190	.2161	.1855	.0534
2	.0489	.0489	.0489	.0489	.0488	.0485	.0444	.0210
3	.0130	.0130	.0130	.0130	.0130	.0130	.0122	.0071
5	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0008
7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

The drawdown in the steady-state region is given by the equation (Jacob, 1946, eq. 15)

$$s = \frac{Q}{2\pi T} K_0(x),$$

where $K_0(x)$ is the zero-order modified Bessel function of the second kind and

$$x = r\sqrt{\frac{K'}{Tb'}}.$$

Data for steady-state conditions can be analyzed using figure 4.3 on plate 1. The drawdowns are plotted versus r and matched to figure 4.3. After choosing a convenient match point with coordinates (s,r) and $(K_0(x),x)$ the parameters are computed from the equations

$$T = \frac{Q}{2\pi s} K_0(x) \text{ and } \frac{K'}{b'} = \frac{xT}{r^2}.$$

Values of $K_0(x)$ from Hantush (1956) are given in table 4.2.

A FORTRAN program for generating type-curve function values of equation 7 is listed in table 4.3. Using the notation $L(u,v)$ of Cooper (1963), the function is evaluated as follows. For $u \geqslant 1$,

$$L(u,v) = \int_u^\infty (1/y) \exp(-y-v^2/y)\, dy = \int_u^\infty f(y)\, dy.$$

This integral is transformed into the form

$$\int_0^\infty e^{-x} \left[\exp\left(-u - \frac{v^2}{x+u}\right) \frac{1}{x+u}\right] dx$$

evaluated by a Gaussian-Laguerre quadrature formula. For $v^2<u<1$,

TABLE 4.2.—*Selected values of $K_0(x)$*

[From Hantush (1956, p. 704)]

N	$x = NX10^{-2}$	$x = NX10^{-1}$	$x = N$
1	4.7212	2.4271	0.4210
1.5	4.3159	2.0300	.2138
2	4.0285	1.7527	.1139
3	3.6235	1.3725	.0347
4	3.3365	1.1145	.0112
5	3.1142	.9244	.0037
6	2.9329	.7775	------
7	2.7798	.6605	------
8	2.6475	.5653	------
9	2.5310	.4867	------

$$L(u,v) = \int_1^\infty f(y)\, dy + \int_u^1 f(y)\, dy.$$

The first integral is evaluated by a Gaussian-Laguerre quadrature formula, as previously described. The second integral is evaluated using a series expansion, as

$$\int_u^1 f(y)dy = s(1) - s(u),$$

where

$$s = \log u \left[\sum_{n=0}^{\infty} \frac{(v^2)^n}{(n!)^2}\right]$$
$$+ \sum_{m=1}^{\infty} \left\{\frac{(-1)^m}{m}\left[u^m - \left(\frac{v^2}{u}\right)^m\right]\left[\sum_{n=0}^{\infty} \frac{(v^2)^n}{(m+n)!n!}\right]\right\}.$$

For $u < 1$ and $u \leqslant v^2$,

$$L(u,v) = 2K_0(2v) - \int_{\frac{v^2}{u}}^{\infty} f(y)\, dy$$

(Cooper, 1963, p. C50),

where K_0 is the zero-order modified Bessel function of the second kind. The integral in the above expression is evaluated by the Gaussian-Laguerre procedure, as described previously.

Input data for this program consist of three cards with the numeric data coded by specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of $1/u$ for which computation is desired, coded in columns 1–10 in format E10.5; the largest value of $1/u$ for which computation is desired, coded in columns 11–20 in format E10.5. The table will include a range of $1/u$ values spanning these two coded values if the span is less than or equal to 12 log cycles. The next two cards contain 12 values of r/B, all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Zero (or blank) coding is permissible in this field, but computation will terminate with the first zero (or blank) value encountered. An example of the output from this program is shown in figure 4.4.

W(U,R/B)

1/U	R/B 0.10E-05	0.30E-05	0.10E-04	0.30E-04	0.10E-03	0.30E-03	0.10E-02	0.30E-02	0.10E-01
0.100E 01	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194	0.2194
0.150E 01	0.3984	0.3984	0.3984	0.3984	0.3984	0.3984	0.3984	0.3984	0.3984
0.200E 01	0.5598	0.5598	0.5598	0.5598	0.5598	0.5598	0.5598	0.5598	0.5598
0.300E 01	0.8289	0.8289	0.8289	0.8289	0.8289	0.8289	0.8289	0.8289	0.8289
0.500E 01	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226	1.2226
0.700E 01	1.5066	1.5066	1.5066	1.5066	1.5066	1.5066	1.5066	1.5066	1.5065
0.100E 02	1.8229	1.8229	1.8229	1.8229	1.8229	1.8229	1.8229	1.8229	1.8227
0.150E 02	2.1964	2.1964	2.1964	2.1964	2.1964	2.1964	2.1964	2.1964	2.1961
0.200E 02	2.4679	2.4679	2.4679	2.4679	2.4679	2.4679	2.4679	2.4679	2.4675
0.300E 02	2.8570	2.8570	2.8570	2.8570	2.8570	2.8570	2.8570	2.8570	2.8564
0.500E 02	3.3547	3.3547	3.3547	3.3547	3.3547	3.3547	3.3547	3.3546	3.3536
0.700E 02	3.6855	3.6855	3.6855	3.6855	3.6855	3.6855	3.6855	3.6854	3.6839
0.100E 03	4.0379	4.0379	4.0379	4.0379	4.0379	4.0379	4.0379	4.0377	4.0356
0.150E 03	4.4401	4.4401	4.4401	4.4401	4.4401	4.4401	4.4400	4.4397	4.4365
0.200E 03	4.7261	4.7261	4.7261	4.7261	4.7261	4.7261	4.7260	4.7257	4.7212
0.300E 03	5.1299	5.1299	5.1299	5.1299	5.1299	5.1299	5.1298	5.1292	5.1226
0.500E 03	5.6394	5.6394	5.6394	5.6394	5.6394	5.6394	5.6393	5.6383	5.6271
0.700E 03	5.9753	5.9753	5.9753	5.9753	5.9753	5.9753	5.9751	5.9737	5.9580
0.100E 04	6.3315	6.3315	6.3315	6.3315	6.3315	6.3315	6.3313	6.3293	6.3069
0.150E 04	6.7367	6.7367	6.7367	6.7367	6.7367	6.7366	6.7363	6.7333	6.6997
0.200E 04	7.0242	7.0242	7.0242	7.0242	7.0242	7.0241	7.0237	7.0197	6.9750
0.300E 04	7.4295	7.4295	7.4295	7.4295	7.4295	7.4294	7.4287	7.4228	7.3561
0.500E 04	7.9402	7.9402	7.9402	7.9402	7.9402	7.9401	7.9389	7.9290	7.8192
0.700E 04	8.2766	8.2766	8.2766	8.2766	8.2766	8.2764	8.2748	8.2609	8.1092
0.100E 05	8.6332	8.6332	8.6332	8.6332	8.6332	8.6330	8.6307	8.6109	8.3983

FIGURE 4.4.—Example of output from program for computing drawdown due to constant discharge from a well in a leaky artesian aquifer.

Solution 5: Constant discharge from a well in a leaky aquifer with storage of water in the confining beds

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain and underlain everywhere by confining beds having hydraulic conductivities K' and K'', thicknesses b' and b'', and storage coefficients S' and S'', respectively, which are constant in space and time.
4. Flow in the aquifer is two dimensional and radial in the horizontal plane and flow in confining beds is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining beds.
5. Conditions at the far surfaces of the confining beds are (fig. 5.1):
   - Case 1. Constant-head plane sources above and below.
   - Case 2. Impermeable beds above and below.
   - Case 3. Constant-head plane source above and impermeable bed below.

Differential equations:

For the upper confining bed

$$\frac{\partial^2 s_1}{\partial z^2} = \frac{S'}{K'b'}\frac{\partial s_1}{\partial t} \qquad (1)$$

For the aquifer

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K'}{T}\frac{\partial}{\partial z}s_1(r,b',t) - \frac{K''}{T}\frac{\partial}{\partial z}s_2(r,b'+b,\ t) = \frac{S}{T}\frac{\partial s}{\partial t} \qquad (2)$$

For the lower confining bed

$$\frac{\partial^2 s_2}{\partial z^2} = \frac{S''}{K''b''}\frac{\partial s_2}{\partial t} \qquad (3)$$

Equations 1 and 3 are, respectively, the differential equations for nonsteady vertical flow in the upper and lower semipervious beds. Equation 2 is the differential equation for nonsteady two-dimensional radial flow in an aquifer with leakage at its upper and lower boundaries.

Boundary and initial conditions:

Case 1: For the upper confining bed

$$s_1(r,z,0)=0 \qquad (4)$$
$$s_1(r,0,t)=0 \qquad (5)$$
$$s_1(r,b',t)=s(r,t) \qquad (6)$$

For the aquifer

$$s(r,0)=0 \qquad (7)$$
$$s(\infty,t)=0 \qquad (8)$$
$$\lim_{r\to 0} r\frac{\partial s(r,t)}{\partial r} = -\frac{Q}{2\pi T} \qquad (9)$$

For the lower confining bed

$$s_2(r,z,0)=0 \qquad (10)$$
$$s_2(r,b'+b+b'',t)=0 \qquad (11)$$
$$s_2(r,b'+b,t)=s(r,t) \qquad (12)$$

Case 2: Same as case 1, with conditions 5 and 11 being replaced, respectively, by

$$\frac{\partial s_1(r,0,t)}{\partial z} = 0 \qquad (13)$$
$$\frac{\partial s_2(r,b'+b+b'')}{\partial z} = 0 \qquad (14)$$

Case 3: Same as case 1, with condition 11 being replaced by condition 14.

Equations 4, 7, and 10 state that initially the drawdown is zero in the aquifer and within each confining bed. Equation 5 states that a plane of zero drawdown occurs at the top of the upper confining bed. Equations 6 and 12 state that, at the upper and lower boundaries of the aquifer, drawdown in the aquifer is equal to drawdown in the confining beds. Equation 8 states that drawdown is small at a large distance from the pumping well. Equation 9 states that, near the pumping well, the flow is equal to the discharge rate. Equation 11 states that a plane of zero drawdown is at the base of the lower confining bed. Equation 13 states that there is no flow across the top of the upper confining bed. Equation 14 states that no flow occurs across the base of the lower confining bed.

Solutions (Hantush, 1960, p. 3716):

I. For small values of time (t less than both $b'S'/10K'$ and $b''S''/10K''$):

$$s = \frac{Q}{4\pi T} H(u,\beta) , \tag{15}$$

where $$u = \frac{r^2S}{4Tt}$$

and $$\beta = \frac{r}{4}\left(\sqrt{\frac{K'S'}{b'TS}} + \sqrt{\frac{K''S''}{b''TS}}\right)$$

$$H(u,\beta) = \int_u^\infty \frac{e^{-y}}{y}\, erfc \frac{\beta\sqrt{u}}{\sqrt{y(y-u)}}\, dy$$

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\, dy .$$

II. For large values of time:

A. Case 1, t greater than both $5b'S'/K'$ and $5b''S''/K''$

$$s = \frac{Q}{4\pi T} W(u\delta_1,\alpha) , \tag{16}$$

where u is as defined previously

and $$\delta_1 = 1 + (S' + S'')/3S,$$

$$\alpha = r\sqrt{\frac{K'/b'}{T} + \frac{K''/b''}{T}}$$

$$W(u,x) = \int_u^\infty \frac{\exp\,(-y-x^2/4y)}{y}\, dy .$$

B. Case 2, t greater than both $10b'S'/K'$ and $10b''S''/K''$

$$s = \frac{Q}{4\pi T} W(u\delta_2) , \tag{17}$$

where $$\delta_2 = 1 + (S' + S'')/S$$

$$W(u) = \int_u^\infty \frac{e^{-y}}{y}\, dy .$$

C. Case 3, t greater than both $5b'S'/K'$ and $10b''S''/K''$

$$s = \frac{Q}{4\pi T} W\left(u\delta_3,\ r\sqrt{\frac{K'/b'}{T}}\right) , \tag{18}$$

where

$$\delta_3 = 1 + (S'' + S'/3)/S$$

and $W(u,x)$ is as defined in case 1.

Comments:

A cross section through the discharging well is shown in figure 5.1. The flow system is actually three-dimensional in such a geometric configuration. However, as stated by Hantush (1960, p. 3713), if the hydraulic conductivity in the aquifer is sufficiently greater than the hydraulic conductivity of the confining beds, flow will be approximately radial in the aquifer and approximately vertical in the confining beds. A complete solution to this flow problem has not been published. Neuman and Witherspoon (1971, p. 250, eq. II–161) developed a complete solution for case 1 but did not tabulate it. Hantush's solutions, which have been tabulated, are solutions that are applicable for small and large values of time but not for intermediate times.

The "early" data (data collected for small values of t) can be analyzed using equation 15. Figure 5.2 on plate 1 shows plots of $H(u,\beta)$ from Lohman (1972, pl. 4). Hantush (1961d) has an extensive tabulation of $H(u,\beta)$, a part of which is given in table 5.1. The corresponding data curves would consist of observed drawdown versus t/r^2. Superposing the data curves on the type curves and matching the two, with graph axes parallel, so that the data curves lie on or between members of the type-curve family and choosing a convenient match point ($H(u,\beta)$, $1/u$), T and S are computed by

$$T = \frac{Q}{4\pi s} H(u,\beta) ,$$

$$S = 4T\frac{t}{r^2}\Big/\frac{1}{u} .$$

If simplifying conditions are applicable, it is possible to compute the product $K'S'$ from the β value. If $K''S''=0$, $K'S'=16\beta^2b'TS/r^2$, and if $K''S''=K'S'$,

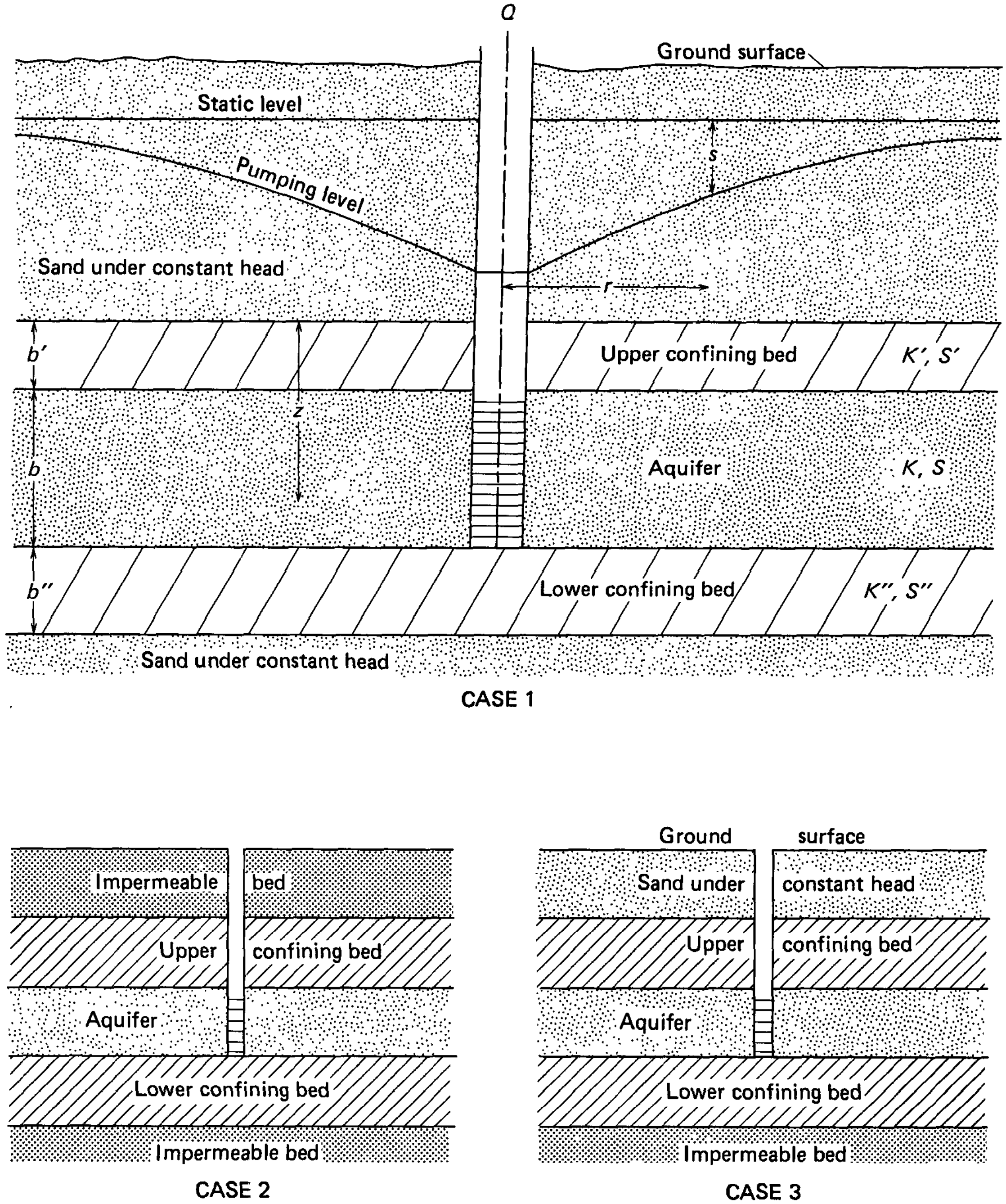

FIGURE 5.1.—Cross sections through discharging wells in leaky aquifers with storage of water in the confining beds, illustrating three different cases of boundary conditions.

$$K'S' = \frac{16\beta^2}{r^2} TS \frac{b'b''}{b'+b''+2\sqrt{b'b''}} .$$

The curves in figure 5.2 are very similar from $\beta=0$ to about $\beta=0.5$. Therefore, the β values in this range are indeterminate. There is also uncertainty in curve matching for all β values because of the fact that it is a family of curves whose shapes change gradually with β. This uncertainty will be increased if the data covers a small range of t values. The problem

TABLE 5.1.—*Values of* H(u,β) *for selected values of* u *and* β

[From Hantush (1961d) Numbers in parentheses are powers of 10 by which the other numbers are multiplied, for example 963(−4) = 0.0963]

u	β							
	0.03	0.1	0.3	1	3	10	30	100
1 × 10^{-9}	12.3088	11.1051	10.0066	8.8030	7.7051	6.5033	5.4101	4.2221
2	11.9622	10.7585	9.6602	8.4566	7.3590	6.1579	5.0666	3.8839
3	11.7593	10.5558	9.4575	8.2540	7.1565	5.9561	4.8661	3.6874
5	11.5038	10.3003	9.2021	7.9987	6.9016	5.7020	4.6142	3.4413
7	11.3354	10.1321	9.0339	7.8306	6.7337	5.5348	4.4487	3.2804
1 × 10^{-8}	11.1569	9.9538	8.8556	7.6525	6.5558	5.3578	4.2737	3.1110
2	10.8100	9.6071	8.5091	7.3063	6.2104	5.0145	3.9352	2.7858
3	10.6070	9.4044	8.3065	7.1039	6.0085	4.8141	3.7383	2.5985
5	10.3511	9.1489	8.0512	6.8490	5.7544	4.5623	3.4919	2.3662
7	10.1825	8.9806	7.8830	6.6811	5.5872	4.3969	3.3307	2.2159
1 × 10^{-7}	10.0037	8.8021	7.7048	6.5032	5.4101	4.2221	3.1609	2.0591
2	9.6560	8.4554	7.3585	6.1578	5.0666	3.8839	2.8348	1.7633
3	9.4524	8.2525	7.1560	5.9559	4.8661	3.6874	2.6469	1.5966
5	9.1955	7.9968	6.9009	5.7018	4.6141	3.4413	2.4137	1.3944
7	9.0261	7.8283	6.7329	5.5346	4.4486	3.2804	2.2627	1.2666
1 × 10^{-6}	8.8463	7.6497	6.5549	5.3575	4.2736	3.1110	2.1051	1.1361
2	8.4960	7.3024	6.2091	5.0141	3.9350	2.7857	1.8074	.8995
3	8.2904	7.0991	6.0069	4.8136	3.7382	2.5984	1.6395	.7725
5	8.0304	6.8427	5.7523	4.5617	3.4917	2.3661	1.4354	.6256
7	7.8584	6.6737	5.5847	4.3962	3.3304	2.2158	1.3061	.5375
1 × 10^{-5}	7.6754	6.4944	5.4071	4.2212	3.1606	2.0590	1.1741	.4519
2	7.3170	6.1453	5.0624	3.8827	2.8344	1.7632	.9339	.3091
3	7.1051	5.9406	4.8610	3.6858	2.6464	1.5965	.8046	.2402
5	6.8353	5.6821	4.6075	3.4394	2.4131	1.3943	.6546	.1685
7	6.6553	5.5113	4.4408	3.2781	2.2619	1.2664	.5643	.1300
1 × 10^{-4}	6.4623	5.3297	4.2643	3.1082	2.1042	1.1359	.4763	963(−4)
2	6.0787	4.9747	3.9220	2.7819	1.8062	.8992	.3287	494(−4)
3	5.8479	4.7655	3.7222	2.5937	1.6380	.7721	.2570	315(−4)
5	5.5488	4.4996	3.4711	2.3601	1.4335	.6252	.1818	166(−4)
7	5.3458	4.3228	3.3062	2.2087	1.3039	.5370	.1412	103(−4)
1 × 10^{-3}	5.1247	4.1337	3.1317	2.0506	1.1715	.4513	.1055	390(−5)
2	4.6753	3.7598	2.7938	1.7516	.9305	.3084	551(−4)	169(−5)
3	4.3993	3.5363	2.5969	1.5825	.8006	.2394	355(−4)	713(−6)
5	4.0369	3.2483	2.3499	1.3767	.6498	.1677	190(−4)	205(−6)
7	3.7893	3.0542	2.1877	1.2460	.5589	.1292	120(−4)	821(−7)
1 × 10^{-2}	3.5195	2.8443	2.0164	1.1122	.4702	955(−4)	695(−5)	274(−7)
2	2.9759	2.4227	1.6853	.8677	.3214	487(−4)	205(−5)	226(−8)
3	2.6487	2.1680	1.4932	.7353	.2491	308(−4)	888(−6)	
5	2.2312	1.8401	1.2535	.5812	.1733	160(−4)	261(−6)	
7	1.9558	1.6213	1.0979	.4880	.1325	982(−5)	106(−6)	
1 × 10^{-1}	1.6667	1.3893	.9358	.3970	966(−4)	552(−5)	365(−7)	
2	1.1278	.9497	.6352	.2452	468(−4)	149(−5)	307(−8)	
3	.8389	.7103	.4740	.1729	281(−4)	592(−6)		
5	.5207	.4436	.2956	.1006	130(−4)	151(−6)		
7	.3485	.2980	.1985	646(−4)	714(−5)	534(−7)		
1 × 1	.2050	.1758	.1172	365(−4)	337(−5)	151(−7)		
2	458(−4)	395(−4)	264(−4)	760(−5)	487(−6)			
3	122(−4)	106(−4)	707(−5)	196(−5)	102(−6)			
5	108(−5)	934(−6)	624(−6)	167(−6)	672(−8)			
7	109(−6)	941(−7)	629(−7)	165(−7)				
1 × 10	391(−8)	339(−8)	227(−8)					
2								
3								
5								
7								

can be avoided, if data from more than one observation well are available, by preparing a composite data plot of s versus t/r^2. This data plot would be matched by adding the constraint that the r values for the different data curves representing each well fall on proportional β curves.

The "late" data (for large values of t) can be analyzed using equations 16, 17, and 18; these equations are forms of summaries 1, $W(u)$, and 4, $L(u, v)$. However, for cases 1 and 3, the late data fall on the flat part of the $L(u,v)$ curves and a time-drawdown plot match would be indeterminate. Thus, only a distance-drawdown match could be used. Drawdown predictions, however, could be made using the $L(u, v)$ curves.

Assumption 5, that no drawdown occurs in the source beds, has been examined by Neuman and Witherspoon (1969a, p. 810, 811) for the situation in which two aquifers are separated by a less permeable bed. This is equivalent to case 3 with $K''=0$ and $S''=0$. They concluded that (1) $H(u,\beta)$, in the asymptotic solution for early times, would not be affected appreciably because the properties of the source bed have a negligible effect on the solution for $Tt/r^2S \leqslant 1.6\beta^2/(r/B)^4$, which is equivalent to $t \leqslant S'b'/10K'$, where $B=\sqrt{Tb'/K'}$; and (2) if $T_s > 100T$, where T_s represents the transmissivity of the source bed, it is probably justified to neglect drawdown in the unpumped aquifer.

Table 5.2 is a listing of a FORTRAN program for computing values of $H(u,\beta)$ for $u \geqslant 10^{-60}$ using a procedure devised and programed by S. S. Papadopulos. Input data for this program consists of three cards. The first card contains the beginning value of $1/u$, coded in columns 1–10, in format E10.5, and the ending (largest) value of $1/u$, coded in columns 11–20, in format E10.5. The next two cards contain 12 values of β, coded in columns 1–10, 11–20, . . . , and 71–80 on the first card and columns 1–10, 11–20, . . . , 31–40 on the second card, all in format E10.5. The function is evaluated as follows (S. S. Papadopulos, written commun., 1975):

$$H(u,\beta) = \int_u^\infty (e^{-y}/y)\ erfc\ (\beta\sqrt{u}/\sqrt{y(y-u)})\ dy$$

$$= \int_u^\infty f\,dy\ ,$$

where f represents the integrand. For $\beta=0$, $H(u,\beta)=W(u)$, where $W(u)$ is the well function of Theis. Because $erfc(x) \leqslant 1$ for $x \geqslant 0$, it follows that $H(u,\beta) \leqslant W(u)$, and for $u>10$, $W(u) \approx 0$ and therefore for $u>10$, $H(u,\beta) \approx 0$. The tables of $H(u,\beta)$ indicate that $H(u,\beta) \approx 0$ for $\beta>1$ and $\beta^2 u>300$. For an arbitrarily small value of u, the integral can be considered as the sum of three integrals

$$\int_u^\infty f\,dy = \int_u^{u_1} f\,dy + \int_{u_1}^{u_2} f\,dy + \int_{u_2}^\infty f\,dy\ ,$$

where $u_2 = (u/2)(1 + \sqrt{1+10^{20}\beta^2/u})$,

and $u_1 = (u/2)(1 + \sqrt{1+0.025\ \beta^2/u})$.

The significance of u_2 and u_1 is that

$$erfc\ (\beta\sqrt{u}/\sqrt{y(y-u)}) \approx 1 \text{ for } u>u_2$$

and

$$erfc\ (\beta\sqrt{u}/\sqrt{y(y-u)}) \approx 0 \text{ for } u<u_1.$$

Therefore,

$$\int_u^{u_1} f\,dy \approx 0\ ,$$

and

$$\int_{u_2}^\infty f\,dy \approx W(u_2),$$

where $W(u_2)$ is the well function of Theis. The function can be evaluated as

$$H(u,\beta) \approx W(u) \text{ for } u > u_2$$

$$H(u,\beta) \approx \int_u^{u_2} f\,dy + W(u_2) \text{ for } u_1 < u < u_2$$

and $$H(u,\beta) \approx \int_{u_1}^{u_2} f\,dy + W(u_2) \text{ for } u < u_1\ .$$

If $u_2 > 10$, then

$$\int_{u_1}^{u_2} f\,dy = \int_{u_1}^{10} f\,dy,\ W(u_2) \approx 0\ .$$

An example of output from this program is shown in figure 5.3.

Solution 6: Constant discharge from a partially penetrating well in a leaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.

H(U,BETA)

1/U | BETA	0.30E-01	0.10E 00	0.30E 00	0.10E 01	0.30E 01
0.100E 02	1.6667	1.3894	0.9358	0.3970	0.0966
0.150E 02	1.9953	1.6531	1.1203	0.5010	0.1379
0.200E 02	2.2308	1.8401	1.2536	0.5812	0.1733
0.300E 02	2.5626	2.1010	1.4435	0.7023	0.2320
0.500E 02	2.9759	2.4228	1.6853	0.8677	0.3214
0.700E 02	3.2428	2.6296	1.8457	0.9836	0.3897
0.100E 03	3.5196	2.8443	2.0164	1.1122	0.4702
0.150E 03	3.8256	3.0826	2.2112	1.2647	0.5717
0.200E 03	4.0369	3.2483	2.3499	1.3767	0.6498
0.300E 03	4.3259	3.4775	2.5459	1.5394	0.7683
0.500E 03	4.6754	3.7598	2.7938	1.7516	0.9305
0.700E 03	4.8969	3.9425	2.9576	1.8953	1.0447
0.100E 04	5.1247	4.1338	3.1317	2.0507	1.1715
0.150E 04	5.3756	4.3486	3.3301	2.2306	1.3225
0.200E 04	5.5488	4.4996	3.4712	2.3602	1.4335
0.300E 04	5.7871	4.7109	3.6704	2.5452	1.5951
0.500E 04	6.0787	4.9747	3.9220	2.7819	1.8062
0.700E 04	6.2665	5.1474	4.0880	2.9396	1.9494
0.100E 05	6.4623	5.3297	4.2643	3.1082	2.1042
0.150E 05	6.6816	5.5361	4.4650	3.3014	2.2837
0.200E 05	6.8353	5.6821	4.6076	3.4394	2.4131
0.300E 05	7.0498	5.8874	4.8087	3.6349	2.5979
0.500E 05	7.3170	6.1454	5.0624	3.8827	2.8344
0.700E 05	7.4915	6.3149	5.2297	4.0467	2.9921
0.100E 06	7.6754	6.4944	5.4072	4.2212	3.1606
0.150E 06	7.8834	6.6983	5.6090	4.4202	3.3538
0.200E 06	8.0304	6.8427	5.7523	4.5617	3.4917
0.300E 06	8.2369	7.0462	5.9544	4.7616	3.6872
0.500E 06	8.4960	7.3024	6.2091	5.0141	3.9351
0.700E 06	8.6662	7.4710	6.3770	5.1807	4.0991
0.100E 07	8.8463	7.6497	6.5549	5.3576	4.2736
0.150E 07	9.0507	7.8528	6.7573	5.5589	4.4726
0.200E 07	9.1955	7.9968	6.9010	5.7018	4.6141
0.300E 07	9.3995	8.1998	7.1034	5.9035	4.8141
0.500E 07	9.6560	8.4554	7.3586	6.1578	5.0666
0.700E 07	9.8249	8.6237	7.5267	6.3255	5.2332
0.100E 08	10.0038	8.8022	7.7049	6.5033	5.4101
0.150E 08	10.2070	9.0050	7.9075	6.7055	5.6114
0.200E 08	10.3512	9.1489	8.0512	6.8490	5.7544
0.300E 08	10.5543	9.3517	8.2539	7.0513	5.9561
0.500E 08	10.8101	9.6072	8.5092	7.3063	6.2104
0.700E 08	10.9785	9.7754	8.6773	7.4744	6.3781
0.100E 09	11.1570	9.9538	8.8556	7.6525	6.5559
0.150E 09	11.3599	10.1566	9.0583	7.8550	6.7581
0.200E 09	11.5039	10.3004	9.2021	7.9988	6.9016
0.300E 09	11.7067	10.5032	9.4048	8.2014	7.1040
0.500E 09	11.9622	10.7586	9.6602	8.4566	7.3590
0.700E 09	12.1305	10.9269	9.8284	8.6248	7.5270
0.100E 10	12.3089	11.1052	10.0067	8.8031	7.7052

FIGURE 5.3.—Example of output from program for computing drawdown due to constant discharge from a well in a leaky aquifer with storage of water in the confining beds.

4. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
5. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
6. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
7. Flow is vertical in the confining bed.
8. The leakage from the confining bed is assumed to be generated within the aquifer so that in the aquifer no vertical flow results from leakage alone.

Differential equation:

$$\partial^2 s/\partial r^2 + 1/r\ \partial s/\partial r + a^2 \partial^2 s/\partial z^2 - sK'/Tb' = S/T\ \partial s/\partial t$$

$$a^2 = K_z/K_r$$

This is the differential equation describing nonsteady radial and vertical flow in a homogeneous aquifer with radial-vertical anisotropy and leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,z,0)=0,\ r\geq 0,\ 0\leq z\leq b \qquad (1)$$
$$s(\infty,z,t)=0,\ 0\leq z\leq b,\ t\geq 0 \qquad (2)$$
$$\partial s(r,0,t)/\partial z=0,\ r\geq 0,\ t\geq 0 \qquad (3)$$
$$\partial s(r,b,t)/\partial z=0,\ r\geq 0,\ t\geq 0 \qquad (4)$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = \begin{cases} 0, & \text{for } 0<z<d \\ -Q/(2\pi K_r(l-d)), & \text{for } d<z<l \\ 0, & \text{for } l<z<b \end{cases} \qquad (5)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen and not by leakage. Equation 5 states that near the pumping well the discharge is distributed uniformly over the well screen and that no radial flow occurs above and below the screen.

Solution:

I. For the drawdown in a piezometer, a solution by Hantush (1964a, p. 350) is given by

$$s = Q/4\pi T\{W(u,\beta) + f(u,ar/b,\beta,d/b,l/b,z/b)\},$$

where

$$W(u,\beta) = \int_u^\infty \frac{e^{-y-\frac{\beta^2}{4y^2}}}{y}\,dy$$

$$u = \frac{r^2 S}{4Tt}$$

$$\beta = \sqrt{\frac{r^2K'}{Tb'}}$$

$$a = \sqrt{K_z/K_r}$$

$$f(u,ar/b,\beta,d/b,l/b,z/b) = 2b/\pi(l-d)\sum_{n=1}^{\infty} l/n(\sin n\pi l/b - \sin n\pi d/b) \cdot\cos(n\pi z/b)W\left(u,\sqrt{\beta^2+(n\pi ar/b)^2}\right).$$

II. For the drawdown in an observation well

$$s = Q/4\pi T\{W(u,\beta) + \bar{f}(u,ar/b,\beta,d/b,l/b,d'/b,l'/b)\},$$

where

$$\bar{f}(u,ar/b,\beta,d/b,l/b,d'/b,l'/b) = 2b^2/\pi^2(l-d)(l'-d') \cdot\sum_{n=1}^{\infty} 1/n^2(\sin n\pi l/b - \sin n\pi d/b) \cdot(\sin n\pi l'/b - \sin n\pi d'/b)W(u,\sqrt{\beta^2+(n\pi ar/b)^2})$$

Comments:

The geometry is shown in figure 6.1. The differential equation and boundary conditions are based on the assumption that vertical flow in the aquifer is caused by partial penetration of the pumping well and not by leakage. Hantush (1967, p. 587) concluded that this assumption is correct if $b\sqrt{K'/Tb'} < 0.1$. The solutions are based on a uniform distribution of flow over the screen of the pumped well. Depending on friction losses within the well, a more realistic assumption might be constant drawdown over

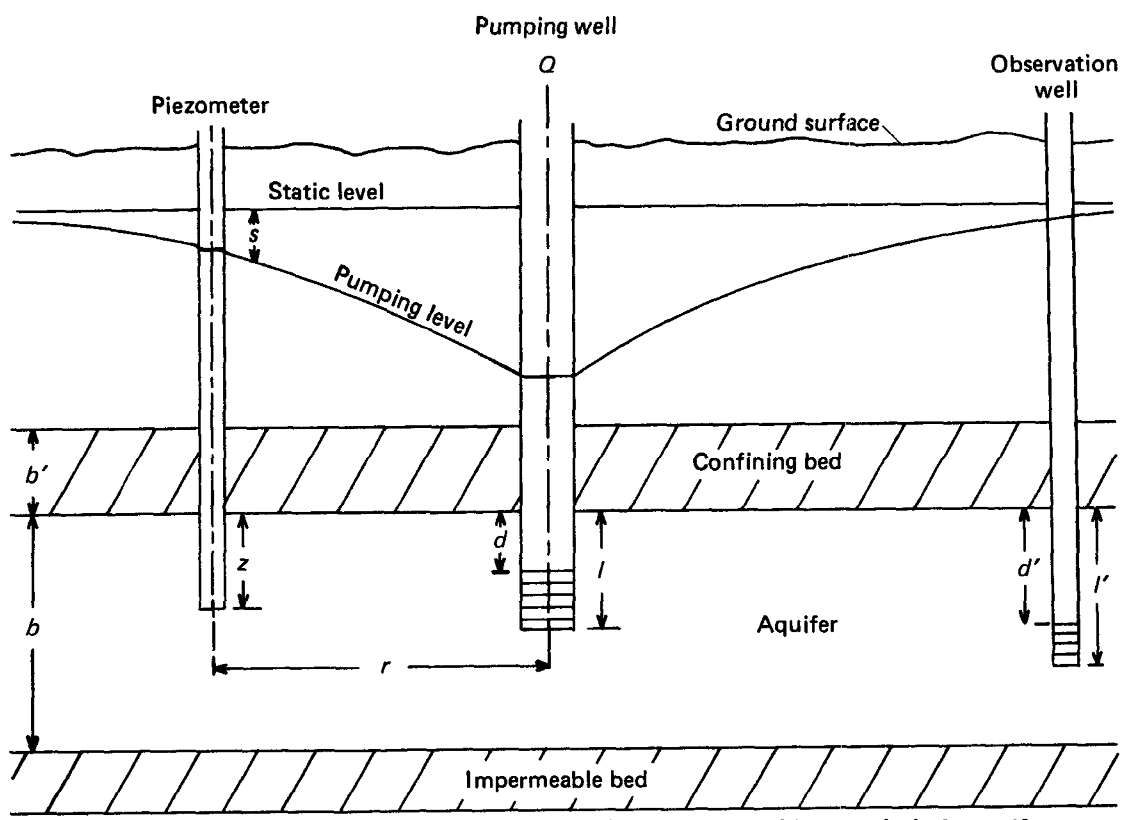

FIGURE 6.1.—Cross section through a discharging well that is screened in part of a leaky aquifer.

the screen of the pumped well; this assumption would imply nonuniform distribution of flow. Hantush (1964a, p. 351) postulates that the actual drawdown at the face of the pumping well will have a value between these two extremes. The solutions should be applied with caution at locations very near the pumped well. The effects of partial penetration are insignificant for $r>1.5\ b/a$ (Hantush, 1964a, p. 350), and the solution is the same for the solution 4.

Because of the large number of variables involved, presentation of a complete set of type curves is impractical. An example, consisting of curves for selected values of the parameters, is shown in figure 6.2 on plate 1. This figure is based on function values generated by a FORTRAN program.

The computer program formulated to compute drawdowns due to pumping a partially penetrating well in a leaky aquifer is listed in table 6.1. Input data to this program consists of cards coded in specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: aquifer thickness (b), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of pumping well screen (l), coded in format F5.1 in columns 6–10; depth, below top of aquifer, to top of pumping well screen (d), coded in format F5.1 in columns 11–15; number of observation wells and piezometers, coded in format I5 in columns 16–20; smallest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 21–30; largest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 31–40. The next two cards contain 12 values of r/B, all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Computation will terminate with the first zero (or blank) value coded. Next is a series of cards, one card per observation well or piezometer, containing: radial distance from the pumped well multiplied by the square root of the ratio of vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of observation well screen (code blank for piezometer), coded in format F5.1, in columns 6–10; depth, below top of aquifer, to top of observation well screen (total depth for a piezometer), coded in format F5.1, in columns 11–15. Output from this program is a table of function values. An example of the output is shown in figure 6.3.

Because most aquifers are anisotropic in the $r-z$ plane, it is generally impractical to use this solution to analyze for the parameters. However, it can be used to predict drawdown if the parameters are determined independently.

W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,Z/B), Z/B= 0.50, SQRT(KZ/KR)*R/B= 0.10, L/B= 0.70, D/B= 0.30

1/U	R/BR 0.10E-05	0.10E-04	0.10E-03	0.10E-02	0.10E-01	0.10E 00	0.10E 01	0.10E 02
0.100E 01	0.5478	0.5478	0.5478	0.5478	0.5478	0.5468	0.4631	0.0001
0.150E 01	0.9901	0.9901	0.9901	0.9901	0.9900	0.9878	0.7872	0.0001
0.200E 01	1.3804	1.3804	1.3804	1.3804	1.3803	1.3764	1.0398	0.0001
0.300E 01	2.0043	2.0043	2.0043	2.0043	2.0042	1.9964	1.3767	0.0001
0.500E 01	2.8381	2.8381	2.8381	2.8381	2.8379	2.8221	1.6931	0.0001
0.700E 01	3.3737	3.3737	3.3737	3.3737	3.3735	3.3499	1.8159	0.0001
0.100E 02	3.9049	3.9049	3.9049	3.9049	3.9046	3.8700	1.8826	0.0001
0.150E 02	4.4488	4.4488	4.4488	4.4488	4.4483	4.3975	1.9094	0.0001
0.200E 02	4.7951	4.7951	4.7951	4.7951	4.7944	4.7291	1.9143	0.0001
0.300E 02	5.2379	5.2379	5.2379	5.2379	5.2369	5.1455	1.9155	0.0001
0.500E 02	5.7539	5.7539	5.7539	5.7539	5.7525	5.6135	1.9155	0.0001
0.700E 02	6.0864	6.0864	6.0864	6.0864	6.0844	5.9001	1.9155	0.0001
0.100E 03	6.4390	6.4390	6.4390	6.4389	6.4363	6.1859	1.9155	0.0001
0.150E 03	6.8411	6.8411	6.8411	6.8411	6.8372	6.4816	1.9155	0.0001
0.200E 03	7.1271	7.1271	7.1271	7.1271	7.1220	6.6669	1.9155	0.0001
0.300E 03	7.5309	7.5309	7.5309	7.5309	7.5233	6.8854	1.9155	0.0001
0.500E 03	8.0404	8.0404	8.0404	8.0403	8.0278	7.0788	1.9155	0.0001
0.700E 03	8.3763	8.3763	8.3763	8.3762	8.3588	7.1556	1.9155	0.0001
0.100E 04	8.7326	8.7326	8.7326	8.7323	8.7076	7.2002	1.9155	0.0001
0.150E 04	9.1377	9.1377	9.1377	9.1373	9.1005	7.2199	1.9155	0.0001
0.200E 04	9.4252	9.4252	9.4252	9.4247	9.3758	7.2239	1.9155	0.0001
0.300E 04	9.8305	9.8305	9.8305	9.8298	9.7568	7.2250	1.9155	0.0001
0.500E 04	10.3412	10.3412	10.3412	10.3400	10.2199	7.2251	1.9155	0.0001
0.700E 04	10.6776	10.6776	10.6776	10.6759	10.5099	7.2251	1.9155	0.0001
0.100E 05	11.0343	11.0343	11.0343	11.0318	10.7990	7.2251	1.9155	0.0001

W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,L'/B,D'/B), L'/B= 0.51, D'/B= 0.49, SQRT(KZ/KR)*R/B= 0.10, L/B= 0.70, D/B= 0.30

1/U	R/BR 0.10E-05	0.10E-04	0.10E-03	0.10E-02	0.10E-01	0.10E 00	0.10E 01	0.10E 02
0.100E 01	0.5477	0.5477	0.5477	0.5477	0.5477	0.5468	0.4631	0.0001
0.150E 01	0.9899	0.9899	0.9899	0.9899	0.9899	0.9876	0.7871	0.0001
0.200E 01	1.3801	1.3801	1.3801	1.3801	1.3801	1.3761	1.0396	0.0001
0.300E 01	2.0038	2.0038	2.0038	2.0038	2.0037	1.9959	1.3764	0.0001
0.500E 01	2.8372	2.8372	2.8372	2.8372	2.8371	2.8213	1.6927	0.0001
0.700E 01	3.3727	3.3727	3.3727	3.3727	3.3725	3.3488	1.8153	0.0001
0.100E 02	3.9037	3.9037	3.9037	3.9037	3.9034	3.8688	1.8821	0.0001
0.150E 02	4.4475	4.4475	4.4475	4.4475	4.4470	4.3962	1.9089	0.0001
0.200E 02	4.7937	4.7937	4.7937	4.7937	4.7930	4.7277	1.9138	0.0001
0.300E 02	5.2365	5.2365	5.2365	5.2365	5.2356	5.1441	1.9150	0.0001
0.500E 02	5.7525	5.7525	5.7525	5.7525	5.7511	5.6122	1.9150	0.0001
0.700E 02	6.0850	6.0850	6.0850	6.0849	6.0830	5.8987	1.9150	0.0001
0.100E 03	6.4376	6.4376	6.4376	6.4375	6.4349	6.1845	1.9150	0.0001
0.150E 03	6.8397	6.8397	6.8397	6.8397	6.8358	6.4802	1.9150	0.0001
0.200E 03	7.1257	7.1257	7.1257	7.1257	7.1206	6.6655	1.9150	0.0001
0.300E 03	7.5295	7.5295	7.5295	7.5295	7.5219	6.8840	1.9150	0.0001
0.500E 03	8.0390	8.0390	8.0390	8.0389	8.0264	7.0775	1.9150	0.0001
0.700E 03	8.3749	8.3749	8.3749	8.3748	8.3574	7.1542	1.9150	0.0001
0.100E 04	8.7312	8.7312	8.7312	8.7309	8.7062	7.1988	1.9150	0.0001
0.150E 04	9.1363	9.1363	9.1363	9.1359	9.0991	7.2185	1.9150	0.0001
0.200E 04	9.4238	9.4238	9.4238	9.4233	9.3743	7.2225	1.9150	0.0001
0.300E 04	9.8291	9.8291	9.8291	9.8284	9.7554	7.2236	1.9150	0.0001
0.500E 04	10.3398	10.3398	10.3398	10.3386	10.2185	7.2237	1.9150	0.0001
0.700E 04	10.6762	10.6762	10.6762	10.6745	10.5085	7.2237	1.9150	0.0001
0.100E 05	11.0329	11.0329	11.0328	11.0304	10.7976	7.2237	1.9150	0.0001

FIGURE 6.3.—Example of output from program for partial penetration in a leaky artesian aquifer.