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Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B3

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

By J. E. Reed

Book 3
APPLICATIONS OF HYDRAULICS

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PREFACE

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TWI 8-B2. Calibration and maintenance of vertical-axis type current meters, by G. F. Smoot and C. E. Novak. 1968. 15 pages.

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SYMBOLS AND DIMENSIONS

[Numbers in parentheses indicate the solutions to which the definition applies. If no number appears, the symbol has only one definition in this report]

Symbol	*Dimension*	*Description*
a	Dimensionless	$\sqrt{K_z/K_r}$.
b	L	Aquifer thickness.
b'	L	Thickness of confining bed (4, 6, 7, 11); specifically the upper confining bed (5).
b''	L	Thickness of lower confining bed.
d	L	Depth from top of aquifer to top of pumped well screen.
d'	L	Depth from top of aquifer to top of observation-well screen.
H	L	Change in water level in well.
H_0	L	Initial head increase in well.
h	L	Change in water level in aquifer.
K	LT^{-1}	Hydraulic conductivity of aquifer.
K_r	LT^{-1}	Hydraulic conductivity of the aquifer in the radial direction.
K_z	LT^{-1}	Hydraulic conductivity of the aquifer in the vertical direction.
K'	LT^{-1}	Hydraulic conductivity of confining bed (4, 6, 7); specifically the upper confining bed (5).
K''	LT^{-1}	Hydraulic conductivity of lower confining bed.
l	L	Depth from top of aquifer to bottom of pumped well screen.
l'	L	Depth from top of aquifer to bottom of observation-well screen.
Q	L^3T^{-1}	Discharge rate.
$Q(t)$	L^3T^{-1}	Discharge rate.
r	L	Radial distance from center of pumping, flowing, or injecting well.
r_c	L	Radius of well casing or open hole in the interval where the water level changes.
r_w	L	Effective radius of well screen or open hole for pumping, flowing, or injecting well.
S	Dimensionless	Storage coefficient.
S_s	L^{-1}	Specific storage of aquifer.
S_s'	L^{-1}	Specific storage of confining beds.
S'	Dimensionless	Storage coefficient of upper-confining bed.
S''	Dimensionless	Storage coefficient of lower confining bed.
s	L	Drawdown in head (change in water level).
s_1	L	Drawdown in upper confining bed.
s_2	L	Drawdown in lower confining bed.
s_w	L	Constant drawdown in discharging well.
T	L^2T^{-1}	Transmissivity.
T_{xx}, T_{xy}, T_{yy}	L^2T^{-1}	Components of the transmissivity tensor in any orthogonal x-, y-axis system.
$T_{\epsilon\epsilon}$, $T_{\eta\eta}$	L^2T^{-1}	Transmissivities along two principal axes, ϵ and η, such that $T_{\epsilon\eta}=0$.
t	T	Time.
t'	Dimensionless	Variable of integration.
u	Dimensionless	$r^2S/4Tt$(2, 6); variable of integration (3, 7, 9).
v	Dimensionless	Variable of integration.
x	Dimensionless	Dummy variable (2, 5); variable of integration (3).
x, y	L	Distances from the pumped well for an arbitrary rectangular coordinate system (10).
y	Dimensionless	Variable of integration (1, 2, 4, 5, 6).
z	L	Depth from top of aquifer, also, specifically, the depth to bottom of a piezometer (2, 6); depth below top of upper confining bed (5).
z	Dimensionless	Dummy variable (10).
α	Dimensionless	Tt/Sr_w^2.
β	Dimensionless	Variable of integration.
Θ	Dimensionless	Angle between x axis and ϵ axis.
ϵ, η	L	Distances from pumped well in a coordinate system colinear with principal axes of transmissivity tensor.
ρ	Dimensionless	r/r_w.
τ	Dimensionless	Tt/Sr_w^2.

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

By J. E. Reed

Abstract

This report presents type curves and related material for 11 conditions of flow to wells in confined aquifers. These solutions, compiled from hydrologic literature, span an interval of time from Theis (1935) to Papadopulos, Bredehoeft, and Cooper (1973). Solutions are presented for constant discharge, constant drawdown, and variable discharge for pumping wells that fully penetrate leaky and nonleaky aquifers. Solutions for wells that partially penetrate leaky and nonleaky aquifers are included. Also, solutions are included for the effect of finite well radius and the sudden injection of a volume of water for nonleaky aquifers. Each problem includes the partial differential equation, boundary and initial conditions, and solutions. Programs in FORTRAN for calculating additional function values are included for most of the solutions.

Introduction

The purpose of this report is to assemble, under one cover and in a standard format, the more commonly used type-curve solutions for confined ground-water flow toward a well in an infinite aquifer. Some of these solutions are only published in several different journals; some of these journals are not readily obtainable. Other solutions which are included in several references (for example, Ferris and others, 1962; Walton, 1962; Hantush, 1964a; Lohman, 1972) are included here for completeness.

The need for a compendium of type curves for aquifer-test analysis was recognized by Robert W. Stallman, who initiated the work on it. However, ill health and the press of other duties prevented him from personally carrying out his concept, but he never ceased to advocate the need for the compendium. Although it is reduced in scope from his original concept, this report should be recognized to be a result of Stallman's foresight and endeavors in the field of ground-water hydrology.

The type-curve method was devised by C. V. Theis (Wenzel, 1942, p. 88) to determine the two unknown parameters, S and T, in the equations

$$s = (Q/4\pi T)W(u)$$

and

$$u = r^2S/(4Tt),$$

where s is the drawdown in water level in response to the pumping rate Q in an aquifer with transmissivity T and storage coefficient S. The distance r from the pumping well, and the elapsed time t since pumping began, combine with S and T to define a dimensionless variable u and corresponding dimensionless response $W(u)$. Briefly, the method consists of plotting a function curve or type curve, such as $(1/u, W(u))$ on logarithmic-scale graph paper, and plotting the time-drawdown (t-s) data on a second sheet having the same scales. This is equivalent to expressing the preceding equations as

$$\log s = \log Q/4\pi T + \log W(u)$$

and

$$\log 1/u = \log t + \log 4T/r^2S.$$

If the two sheets are superimposed and matched, keeping coordinate axes parallel, as shown in figure 0.1, the respective coordinate

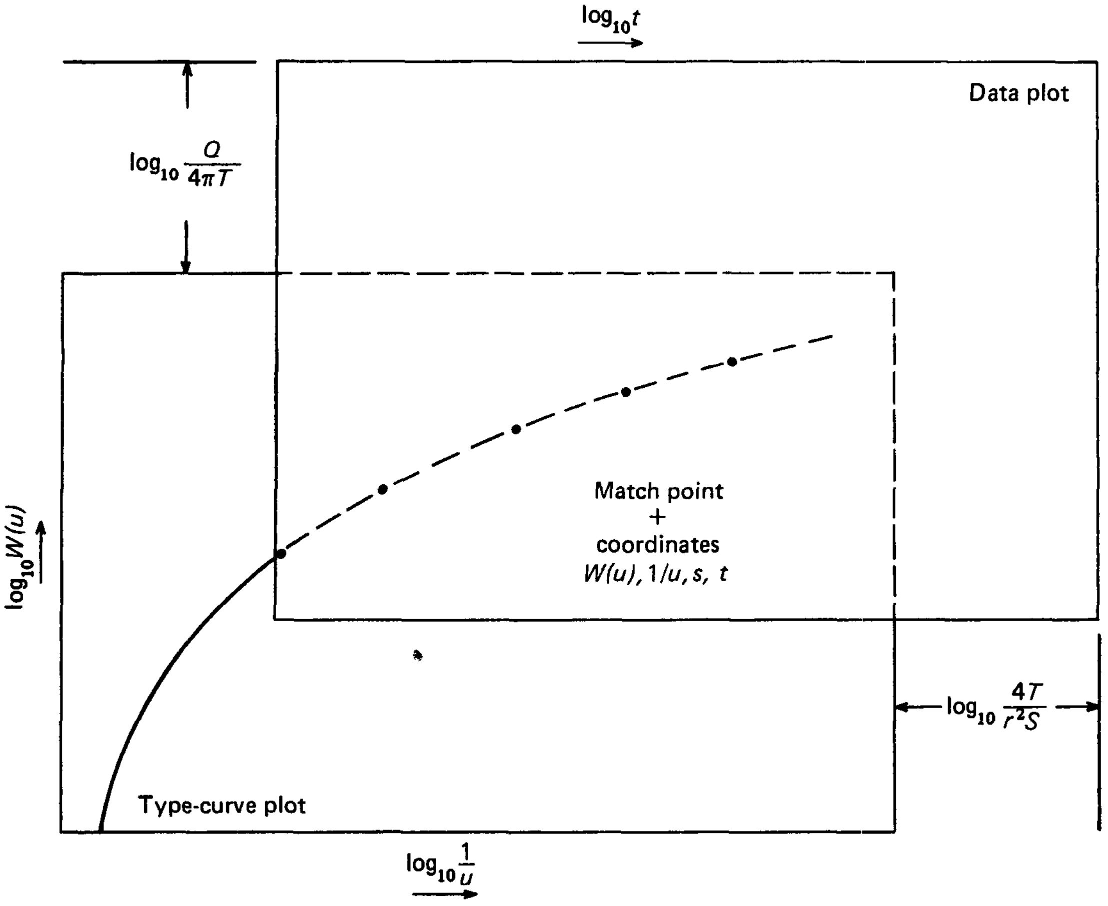

FIGURE 0.1.—Relation of $1/u, W(u)$ type curve and t, s data plot. Modified from Stallman (1971, p. 5, fig. 1).

axes will be related by constant factors: $s/W(u)=C_1$ and $t/(1/u)=C_2$. The values of these two constants are

$$C_1=Q/(4\pi T)$$

and

$$C_2=r^2S/(4T).$$

Thus, a common match point for the two curves may be chosen, and the four coordinate points—$W(u)$, $1/u$, s, and t—recorded for the common match point. T can be obtained from the equation $T=QW(u)/(4\pi s)$, and then S can be solved from the equation $S=4Tut/r^2$, where $W(u)$, $1/u$, s, and t are the match-point values.

It is apparent that the type curves, and data, can be plotted in several ways. That is, the function curve, using $W(u)$ as an example, could be plotted as $(u,W(u))$ with corresponding data plots of $(1/t,s)$ or $(r^2/t,s)$; or could be plotted as $(1/u,W(u))$ with corresponding data plots of (t,s) or $(t/r^2,s)$. The type-curve method is covered more fully by Ferris, Knowles, Brown, and Stallman (1962, p. 94).

The type curves presented in this report are shown on two different plots. One plot has both logarithmic scales with 1.85 inches per log-cycle, such as K and E 467522.[1] The other plot is arithmetic-logarithmic scale with the logarithmic scale 2 inches per log-cycle and the arithmetic scale with divisions at multiples of 0.1, 0.5, and 1.0 inches, such as K and E 466213.

Other methods exist for analysis of aquifer-test data. Among them are methods based on plots of data on semi-log paper, developed by

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[1] The use of brand names in this report is for identification purposes only and does not imply endorsement by the U S Geological Survey

Jacob (Ferris and others, 1962, p. 98) and by Hantush (1956, p. 703). These methods are useful, but they are beyond the scope of this report.

Aquifer tests deal with only one component of the natural flow system. The isolation of the effects of one stress upon the system is based upon the technique of superposition. This technique requires that the natural flow system can be approximated as a linear system, one in which total flow is the addition of the individual flow components resulting from distinct stresses.

The use of the principle of superposition is implied in most aquifer-test analyses. The term "superposition," as here applied, is derived from the theory of linear differential equations. If the partial-differential equation is linear (in the dependent variable and its derivatives), two or more solutions, each for a given set of boundary and initial conditions, can be summed algebraically to obtain a solution for the combined conditions. For instance, consider a situation (fig. 0.2) where a well has been pumping for some time at a constant rate Q_0, and the drawdown trend for that pumping rate has been established. Assume that the pumping rate increases by some amount ΔQ at some time t_1. Then the drawdown for that step incrase in rate will be the change in drawdown from that occurring due to the pumpage Q_0.

Programs, written in FORTRAN, for calculating additional function values are included for most of the solutions. Some of the type-curve solutions would require an unreasonably long tabulation to include all the possible combinations of parameters. An alternative to a tabulation is the computer program that can calculate type-curve values for the parameters desired by the user. The programs could be easily modified to calculate aquifer response to more than one well, such as well fields or image-well systems (Ferris and others, 1962, p. 144). The programs have been tested and are probably reasonably free from error. However, because of the large number of possible parameter combinations, it was possible to test only a sample of possible parameter values. Therefore, errors might occur in future use of these programs.

"An aquifer test is a controlled field experiment made to determine the hydraulic properties of water-bearing and associated rocks" (Stallman, 1971). The areal variability of hydraulic properties in an aquifer limits aquifer tests to integrating these properties within the

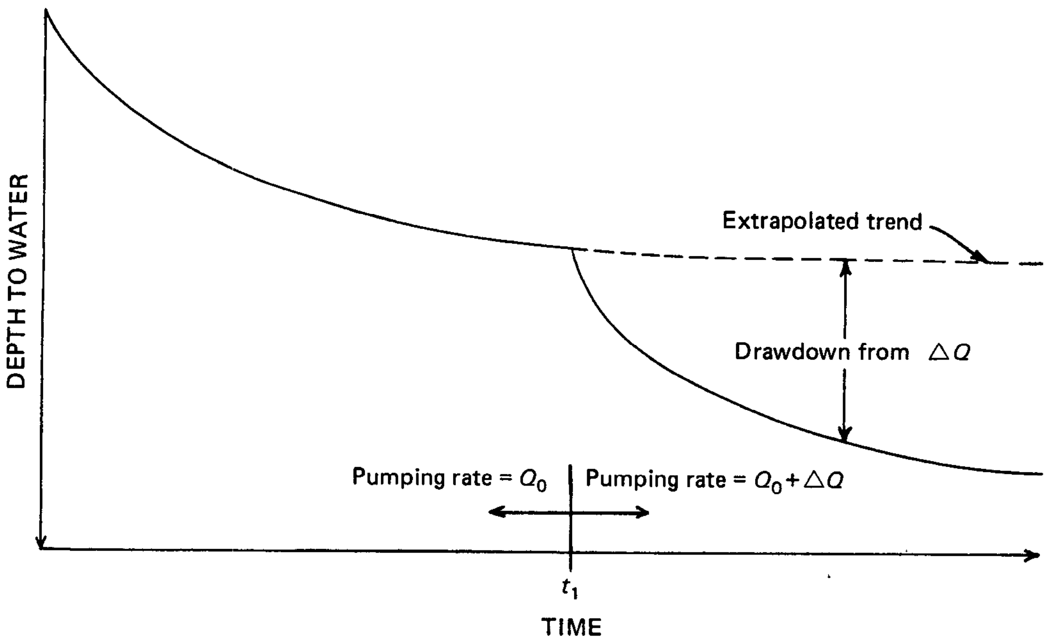

FIGURE 0.2.—The application of the principle of superposition to aquifer tests.

cone of depression produced during the test. Aquifer-test solutions are based on idealized representations of the aquifer, its boundaries, and the nature of the stress on the aquifer. The type-curve solutions presented in this report all have certain assumptions in common. The common assumptions are that the aquifer is horizontal and infinite in areal extent, that water is confined by less permeable beds above and below the aquifer, that the formation parameters are uniform in space and constant in time, that flow is laminar, and that water is released from storage instantaneously with a decline in head. Also implicit is the assumption that hydraulic potential or head is the only cause of flow in the system and that thermal, chemical, density, or other forces are not affecting flow. In addition to these common assumptions are special assumptions that characterize each solution summary. An important first step in aquifer-test analysis is deciding which simplified representations most closely match the usually complex field conditions.

Generally the best start in the analysis of aquifer-test data is with the most general set of type curves that apply to the situation, keeping in mind limitations of the method and effects that cause departures from the theoretical results. For example, the most general set of type curves for constant discharge presented in this report is for leaky aquifers with storage of water in the confining beds, *solution* 5. This includes, as a limiting case, the curve for a nonleaky aquifer. The most severe limitation on this set of curves is that they apply only at early times, as specified in *solution* 5.

Some of the effects that cause departure from the theoretical curves are partial penetration, finite well radius, and variable discharge for the pumped well. The effects of partial penetration must be considered when $r/b<1.5$, and because vertical-horizontal anisotropy is probably a common condition, these effects should be considered for $r/b<10$. The effect of finite well radius should be considered for early times, as specified in *solution* 8. The effects of variable discharge depend upon the manner of the variation. A change in discharge is more important if the change is monotonic, either continually increasing or decreasing. This fact is shown by the type curves for *solution* 11, where a monotonic change of 10 percent caused a significant departure from the Theis curve. If the discharge variation consists of random "noise" about a constant discharge, a 10-percent variation is not significant. The most general set of type curves for tests on flowing wells is *solution* 7, for leaky aquifers, which includes nonleaky aquifers as a limiting case. The only set of curves for slug tests is given in *solution* 9.

A recurring problem in type-curve solution for unknown hydrologic parameters is that of nonuniqueness. That is, function curves for different parameter values sometimes have similar shapes. An example of this is given by Stallman (1971, p. 19 and fig. 6). He indicated that the selection of the conceptual model is very important in interpreting the test results. Equally important is adequate testing of the conceptual model. Corroboration of the conceptual model is indicated by similar results for hydrologic parameters from data collected at varying distances from the pumped well, depths within the aquifer, and at different observation times. However, proof of suitability of the conceptual model ultimately rests on field investigations and not on curve matching.

As an example of similar curve shapes for different situations, consider the case of constant discharge in a nonleaky aquifer with exponentially varying thickness. The thickness, b, is equal to $b_0\exp[-2(X-X_0)/a]$, where b_0 and X_0 are the thickness and X-coordinate, respectively, at the site of the discharging well and a is a parameter. The drawdown for this situation is given by Hantush (1962, p. 1529):

$$s=(Q/4\pi Kb_0)\exp(r/a\cos\Theta)\,W(u,r/a),$$

where

$$W(u,\beta)=\int_u^\infty(\exp(-y-\beta^2/4y)/y)\,dy,$$

$$u=r^2S_s/4Kt,$$

Q is the discharge, r is the distance from the discharging well, Θ is the angle, with apex at the discharging well, between the observation well and the positive X-axis, K is the hydraulic conductivity of the aquifer, and S_s is the specific storage coefficient of the aquifer. This solution is similar to the equation describing drawdown in a leaky artesian aquifer (Hantush, 1956, p. 702), which is

$$s = (Q/4\pi T)\ W(u, r/B),$$

with $T = Kb$, $B = \sqrt{Tb'/K'}$, and b' and K' are the thickness and hydraulic conductivity, respectively, of the leaky confining bed. The other symbols are used as above.

These two functions have the same shape when plotted on logarithmic paper, and drawdown resulting from one function could be matched to a type curve of the other function. Suppose, as an example, that the "observed data" are described by the function for the aquifer with exponentially changing thickness. Suppose, also, that the hydrologist is unaware of the variation in thickness and that the family of type curves for leaky aquifers without storage in the confining beds, *solution* 4, has been chosen for analysis of the "observed data." Matching the data plots to the type curves and solving for unknown parameters by the methods suggested in *solution* 4 gives for the ratio of K_a, the apparent hydraulic conductivity, to K, the true hydraulic conductivity, $K_a/K = \exp((r/a)\cos\Theta)$. The ratio would be close to one only in the vicinity of the discharging well. The diffusivity, K/S_s, would be determined correctly, but the apparent specific storage coefficient would have the same percentage error as the apparent hydraulic conductivity. Most important of all, the erroneous conclusion would be that the aquifer is leaky, with leakage parameter $B = \sqrt{Kbb'/K'} = a$. This somewhat contrived example illustrates a principle in the interpretation of aquifer-test data. Conclusions about the hydrologic constraints on the response of the aquifer to pumping should not be based on the shape of the data curves. Inferences may be made from these curves, but they must be verified by other hydrologic and geologic data. Therefore, proof of the suitability of the conceptual model must come from field investigations.

Many of the old reports of the U.S. Geological Survey contain references to the terms "coefficient of transmissibility" and "field coefficient of permeability." These terms, which were expressed in inconsistent units of gallons and feet, have been replaced by transmissivity and hydraulic conductivity (Lohman and others, 1972, p. 4 and p. 13). Transmissivity and hydraulic conductivity are not solely properties of the porous medium; they are also determined by the kinematic viscosity of the liquid, which is a function of temperature. Field determinations of transmissivity or hydraulic conductivity are made at prevailing field temperatures, and no corrections for temperature are made.

Summaries of Type-Curve Solutions for Confined Ground-Water Flow Toward a Well in an Infinite Aquifer

Solution 1: Constant discharge from a fully penetrating well in a nonleaky aquifer (Theis equation)

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\partial^2 s/\partial r^2 + (1/r)\ (\partial s/\partial r) = (S/T)(\partial s/\partial t)$$

Boundary and initial conditions:

$$s(r,0) = 0,\ r \geqslant 0 \tag{1}$$

$$s(\infty,t) = 0,\ t \geqslant 0 \tag{2}$$

$$Q = \begin{cases} 0, t<0 \\ \text{constant} >0,\ t \geqslant 0 \end{cases} \tag{3}$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T},\quad t \geqslant 0 \tag{4}$$

Equation 1 states that initially drawdown is zero everywhere in the aquifer. Equation 2 states that the drawdown approaches zero as the distance from the well approaches infinity. Equation 3 states that the discharge from the well is constant throughout the pumping period. Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.

Solution (Theis, 1935):

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y}}{y}\,dy$$

$$u = \frac{r^2S}{4Tt},$$

where

$$\int_u^\infty \frac{e^{-y}}{y}\,dy = W(u) = -0.577216 - \log_e u + u$$

$$-\frac{u^2}{2!\,2} + \frac{u^3}{3!\,3} - \frac{u^4}{4!\,4} + \ldots\ldots\ldots$$

Comments:

Assumptions made are applicable to artesian aquifers (fig. 1.1). However, the solution may be applied to unconfined aquifers if drawdown is small compared with the saturated thickness of the aquifer and if water in the sediments through which the water table has fallen is discharged instantaneously with the fall of the water table. According to assumption 2, this solution does not consider the effect of the change in storage within the pumping well. Assumption 2 is acceptable if

$$t > 2.5\times10^2 r_c^2/T$$

(Papadopulos and Cooper, 1967, p. 242), where r_c is the radius of the well casing in the interval over which the water-level declines, and other symbols are as defined previously. Figure 1.2 on plate 1 is a logarithmic graph of $W(u)=4\pi sT/Q$ plotted on the vertical coordinates versus $1/u=4Tt/(r^2S)$ plotted on the horizontal coordinates. The test data should be plotted with s on the vertical coordinates and corresponding values of t or t/r^2 on the horizontal coordinates.

Values of $W(u)$ for u between 0 and 170 may be computed by using subroutine EXPI of the IBM System/360 Scientific Subroutine Package. Table 1.1 gives values of $W(u)$ for selected values of $1/u$ between 1×10^{-1} and 9×10^{14}, as calculated by this subroutine.

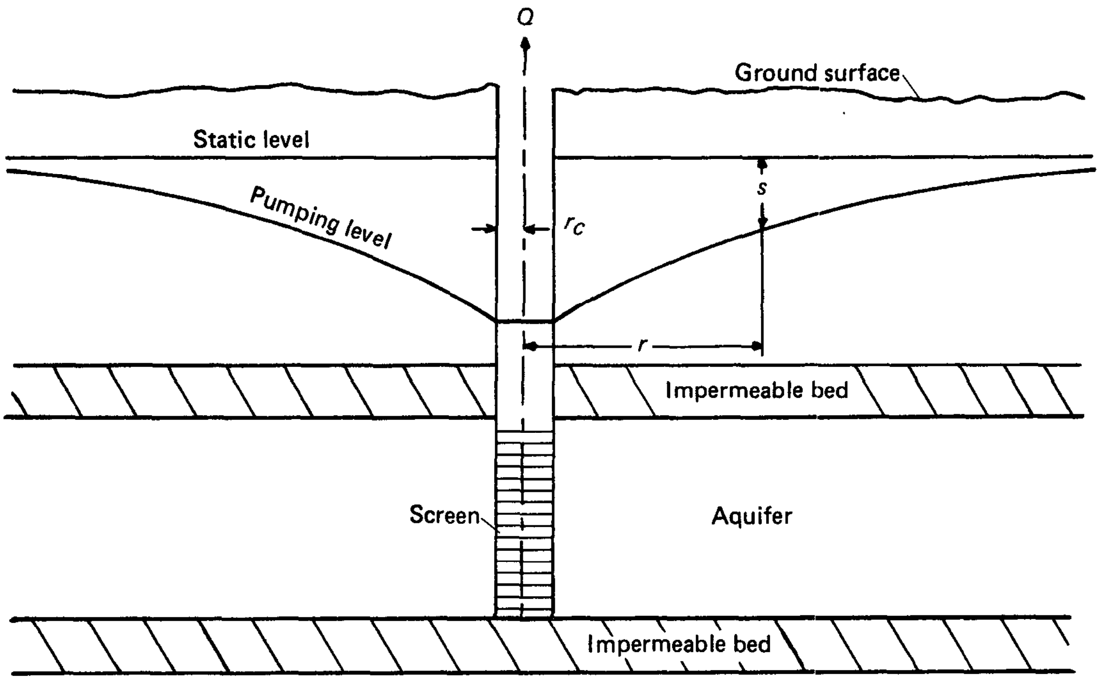

FIGURE 1.1.—Cross section through a discharging well in a nonleaky aquifer.

TABLE 1.1.—*Values of Theis equation* W(u) *for values of* 1/u

$1/u$	$1/u \times 10^{-1}$	1	10	10^2	10^3	10^4	10^5	10^6
1 0	[1]0 00000	0 21938	1 82292	4 03793	6 33154	8 63322	10 93572	13.23830
1.2	00003	29255	1 98932	4.21859	6 51369	8.81553	11.11804	13.42062
1 5	.00017	.39841	2.19641	4.44007	6.73667	9.03866	11.34118	13 64376
2 0	.00115	55977	2.46790	4.72610	7.02419	9.32632	11.62886	13.93144
2 5	00378	.70238	2.68126	4.94824	7 24723	9 54945	11 85201	14.15459
3 0	00857	82889	2.85704	5.12990	7.42949	9 73177	12.03433	14 33691
3.5	01566	94208	3 00650	5 28357	7.58359	9.88592	12.18847	14.49106
4 0	02491	1.04428	3 13651	5 41675	7.71708	10.01944	12.32201	14.62459
5 0	04890	1.22265	3 35471	5.63939	7.94018	10.24258	12 54515	14.84773
6 0	.07833	1 37451	3 53372	5 82138	8 12247	10.42490	12.72747	15.03006
7 0	11131	1.50661	3.68551	5.97529	8 27659	10.57905	12.88162	15 18421
8 0	14641	1 62342	3 81727	6 10865	8.41011	10.71258	13.01515	15.31774
9.0	18266	1 72811	3 93367	6.22629	8.52787	10.83036	13.13294	15.43551

$1/u$	$1/u \times 10^7$	10^8	10^9	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}
1.0	15 54087	17.84344	20.14604	22.44862	24 75121	27 05379	29.35638	31 65897
1.2	15.72320	18 02577	20.32835	22.63094	24.93353	27 23611	29.53870	31.84128
1.5	15.94634	18.24892	20.55150	22 85408	25 15668	27 45926	29 76184	32.06442
2 0	16 23401	18.53659	20 83919	23 14177	25 44435	27 74693	30.04953	32.35211
2 5	16.45715	18 75974	21.06233	23.36491	25.66750	27.97008	30 27267	32.57526
3 0	16.63948	18 94206	21 24464	23.54723	25 84982	28.15240	30.45499	32.75757
3 5	16 79362	19.09621	21.39880	23.70139	26.00397	28.30655	30 60915	32.91173
4 0	16.92715	19 22975	21.53233	23.83492	26 13750	28.44008	30.74268	33 04526
5.0	17 15030	19.45288	21 75548	24 05806	26 36064	28 66322	30.96582	33.26840
6 0	17 33263	19 63521	21 93779	24.24039	26 54297	28.84555	31.14813	33.45071
7.0	17.48677	19 78937	22 09195	24.39453	26.69711	28 99969	31.30229	33.60487
8 0	17.62030	19.92290	22.22548	24.52806	26.83064	29.13324	31 43582	33.73840
9.0	17 73808	20.04068	22.34326	24.64584	26.94843	29 25102	31 55360	33.85619

[1] Value shown as 0 00000 is nonzero but less than 0 000005.

Solution 2: Constant discharge from a partially penetrating well in a nonleaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.
4. Aquifer is not leaky.
5. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + a^2\frac{\partial^2 s}{\partial z^2} = \frac{S}{T}\frac{\partial s}{\partial t}$$

$$a^2 = K_z/K_r$$

This is the differential equation for nonsteady radial and vertical flow in a homogeneous confined aquifer with radial-vertical anisotropy.

Boundary and initial conditions:

$$s(r, z, 0) = 0,\ r \geqslant 0,\ 0 \leqslant z \leqslant b \qquad (1)$$

$$s(\infty, z, t) = 0,\ t \geqslant 0 \qquad (2)$$

$$\partial s(r, 0, t)/\partial z = 0,\ r \geqslant 0,\ t \geqslant 0 \qquad (3)$$

$$\partial s(r, b, t)/\partial z = 0,\ r \geqslant 0,\ t \geqslant 0 \qquad (4)$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = \begin{cases} 0, & 0 < z < d \\ -Q/(2\pi K_r(l-d)), & d < z < l \\ 0, & l < z < b \end{cases} \qquad (5)$$

Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that the drawdown approaches zero as the distance from the pumped well approaches infinity. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen, and not by leakage. Equation 5 states that near the pumping well the flow is radial, that the flow toward the well is equal to its discharge, that the discharge is distributed uniformly over the well screen, and that no radial flow occurs above and below the screen.

Solution:

I. For the drawdown in a piezometer, a solution by Hantush (1961a, p. 85, and 1964a, p. 353) is given by

$$s = \frac{Q}{4\pi T}\left[W(u) + f\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right], \qquad (6)$$

where

$$W(u) = \int_u^\infty \frac{e^{-y}}{y}\,dy$$

and

$$f\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right) = \frac{2b}{\pi(l-d)}\sum_{n=1}^{\infty}\frac{1}{n}\cdot$$

$$\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)\cos\frac{n\pi z}{b}\ W\left(u, \frac{n\pi ar}{b}\right) \qquad (7)$$

$$W(u, x) = \int_u^\infty (\exp(-y - x^2/4y)/y)\,dy$$

$$u = \frac{r^2 S}{4Tt}$$

$$a = \sqrt{K_z/K_r}$$

An alternate form of this solution for $a = 1$ is given by Hantush (1961a, p. 85):

$$s = \frac{Qb}{8\pi T(l-d)}\left[M\left(u, \frac{l+z}{r}\right) + M\left(u, \frac{l-z}{r}\right)\right.$$

$$+ f'\left(u, \frac{b}{r}, \frac{l}{r}, \frac{z}{r}\right) - M\left(u, \frac{d+z}{r}\right) - M\left(u, \frac{d-z}{r}\right)$$

$$\left. - f'\left(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}\right)\right], \qquad (8)$$

in which

$$f'\left(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}\right) = \sum_1^\infty\left[M\left(u, \frac{2nb+x+z}{r}\right)\right.$$

$$- M\left(u, \frac{2nb-x-z}{r}\right) + M\left(u, \frac{2nb+x-z}{r}\right)$$

$$\left. - M\left(u, \frac{2nb-x+z}{r}\right)\right] \qquad (9)$$

and

$$M(u,\beta)=\int_u^\infty \frac{e^{-y}}{y}\, erf(\beta\sqrt{y})\, dy$$

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\, dy.$$

II. For the drawdown in an observation well (Hantush, 1961a, p. 90, and 1964a, p. 353),

$$s = \frac{Q}{4\pi T}\left[W(u) + \bar{f}\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right)\right], \quad (10)$$

where $W(u)$ is as defined previously and

$$\bar{f}\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right) = \frac{2b^2}{\pi^2(l-d)(l'-d')}$$

$$\cdot \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)$$

$$\cdot\left(\sin\frac{n\pi l'}{b} - \sin\frac{n\pi d'}{b}\right) W\left(u, \frac{n\pi ar}{b}\right), \quad (11)$$

where $W(u,x)$ and u are as defined previously.

Comments:

Assumptions apply to conditions shown in figure 2.1. The effects of partial penetration need to be considered for $ar/b<1.5$. There must be a type curve for each value of ar/b, d/b, l/b, and either z/b for piezometer, or l'/b and d'/b for observation wells. Because the number of possible type curves is large, only samples of curves for selected values of the parameters are shown in figure 2.2 on plate 1.

For large values of time, that is, for $t>b^2S/(2a^2T)$ or $t>bS/(2K_z)$, the effects of partial penetration are constant in time, and

$$W\left(u, \frac{n\pi ar}{b}\right)$$

can be approximated by

$$2K_0\left(\frac{n\pi ar}{b}\right)$$

(Hantush, 1961a, p. 92). $K_0(x)$ is the modified Bessel function of the second kind of order zero.

Equation 6 then becomes

$$s = \frac{Q}{4\pi T}\, W(u) + \delta s = \frac{Q}{4\pi T}\left[W(u) + f_s\right],$$

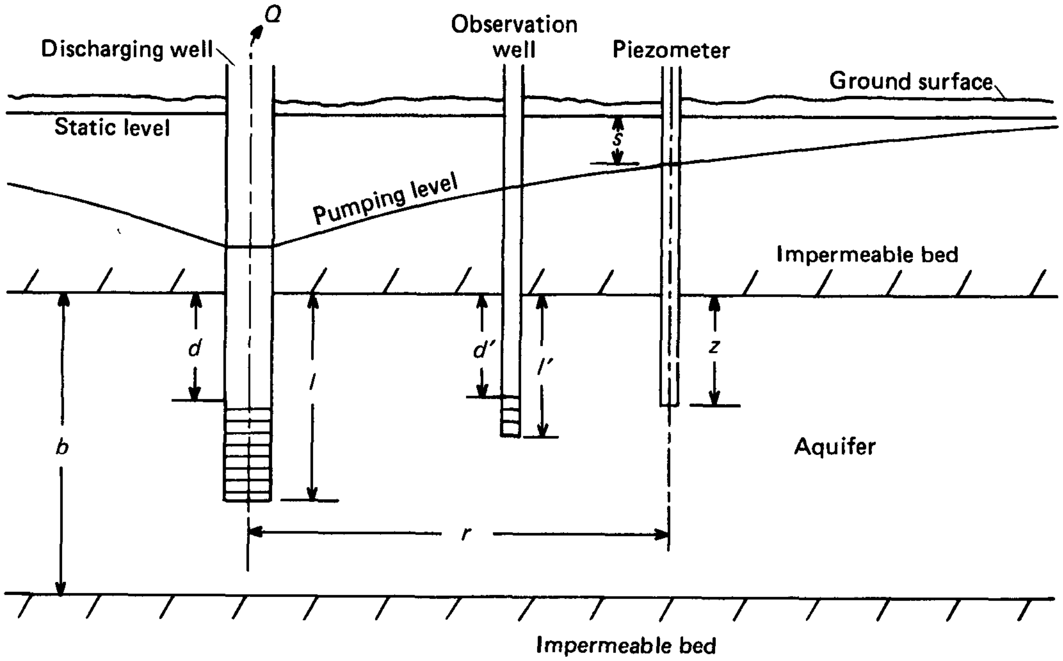

FIGURE 2.1.—Cross section through a discharging well that is screened in a part of a nonleaky aquifer.

where $$\partial s = \frac{Q}{4\pi T} f_s,$$

and f_s is given in equation 7

with $W\left(u, \frac{n\pi ar}{b}\right)$ replaced by $2K_0\left(\frac{n\pi ar}{b}\right)$.

Figure 2.3 shows plots of f_s as tabulated by Weeks (1969, p. 202–207). In using these curves, it should be noted that f_s for a given r, b, and z_1, l_1, d_1 is equal to f_s for the same r, b, and $z_2=b-z_1$, $l_2=b-d_1$, and $d_2=b-l_1$. Figure 2.3 can be used to find f_s by interpolation and then constructing type curves of $W(u)+f_s$ in the manner described by Weeks (1964, p. D195).

For small values of time

$$t < \frac{(2b-l-z)^2 S}{20T}$$

(Hantush, 1961b, p. 172), equation 8 can be approximated by

$$s = \frac{Qb}{8\pi T(l-d)}\left[M\left(u, \frac{l+z}{r}\right) - M\left(u, \frac{d+z}{r}\right) + M\left(u, \frac{l-z}{r}\right) - M\left(u, \frac{d-z}{r}\right)\right].$$

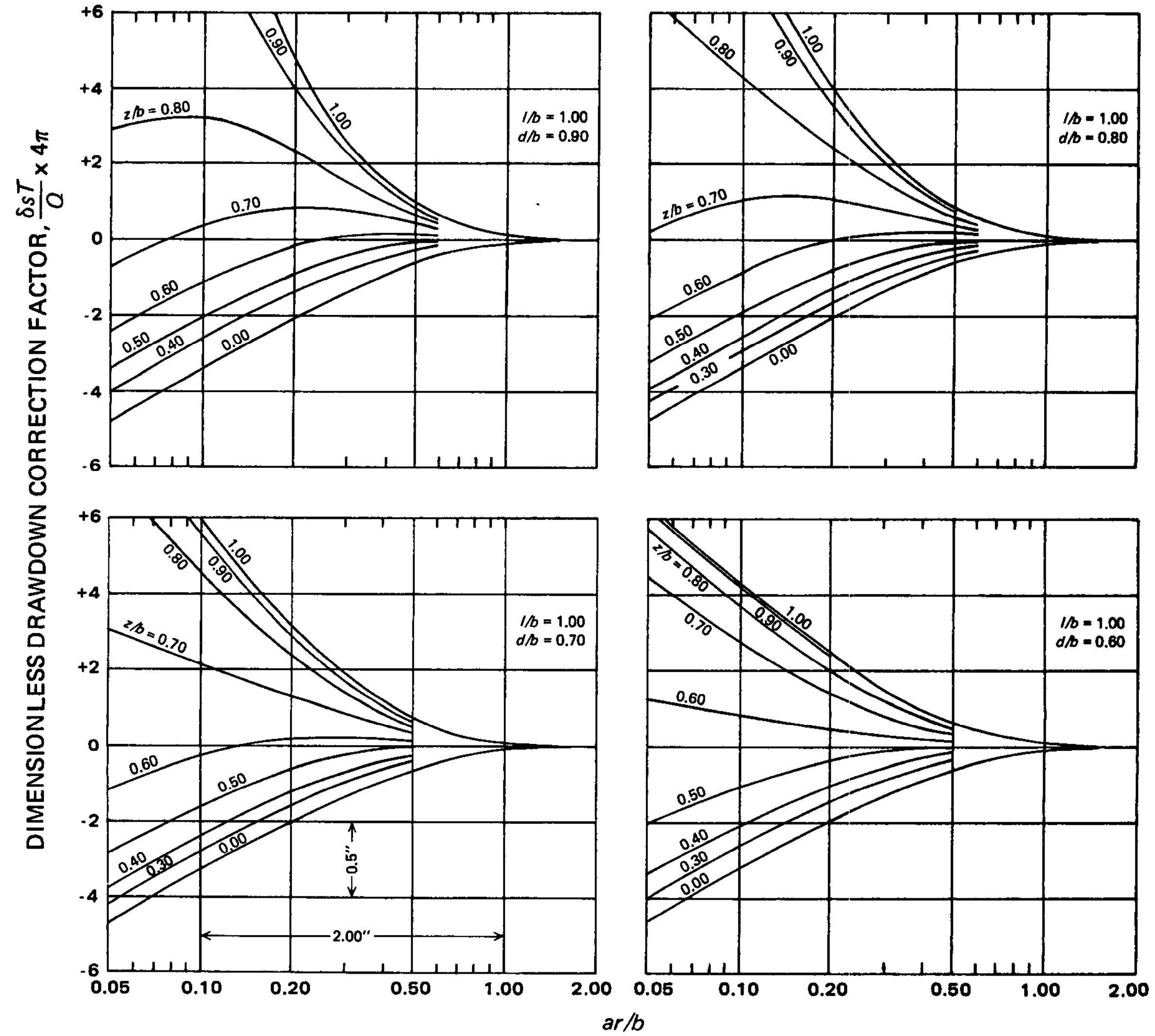

FIGURE 2.3.—The drawdown correction factor f_s versus ar/b, from tables of Weeks (1969).

An extensive table of $M(u,\beta)$ has been prepared by Hantush (1961c).

Although r/b for a given observation well probably would be known, however, the conductivity ratio a^2 would not be. Thus, it would not be known which ar/b curve should be matched. In other words, not only T and S, but also the conductivity ratio a^2 must be determined. A criterion for determining the match between data curves and type curves is that the values of ar/b for different observation wells should all indicate the same "a". Plotting the drawdown data for several observation wells on a single t/r^2 plot and matching to sets of type curves, a different set for each "a", is a useful approach.

Figure 2.2 was prepared from data calculated by the FORTRAN program listed in table 2.1. This program computes "s" from either equation 6 or 10, depending on the input data. The input data consist of cards containing the parameters coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the aquifer thickness (b), coded in columns 1–5, in format F5.1; the depth to bottom of pumped well screen (l), coded in columns 6–10, in format F5.1; the

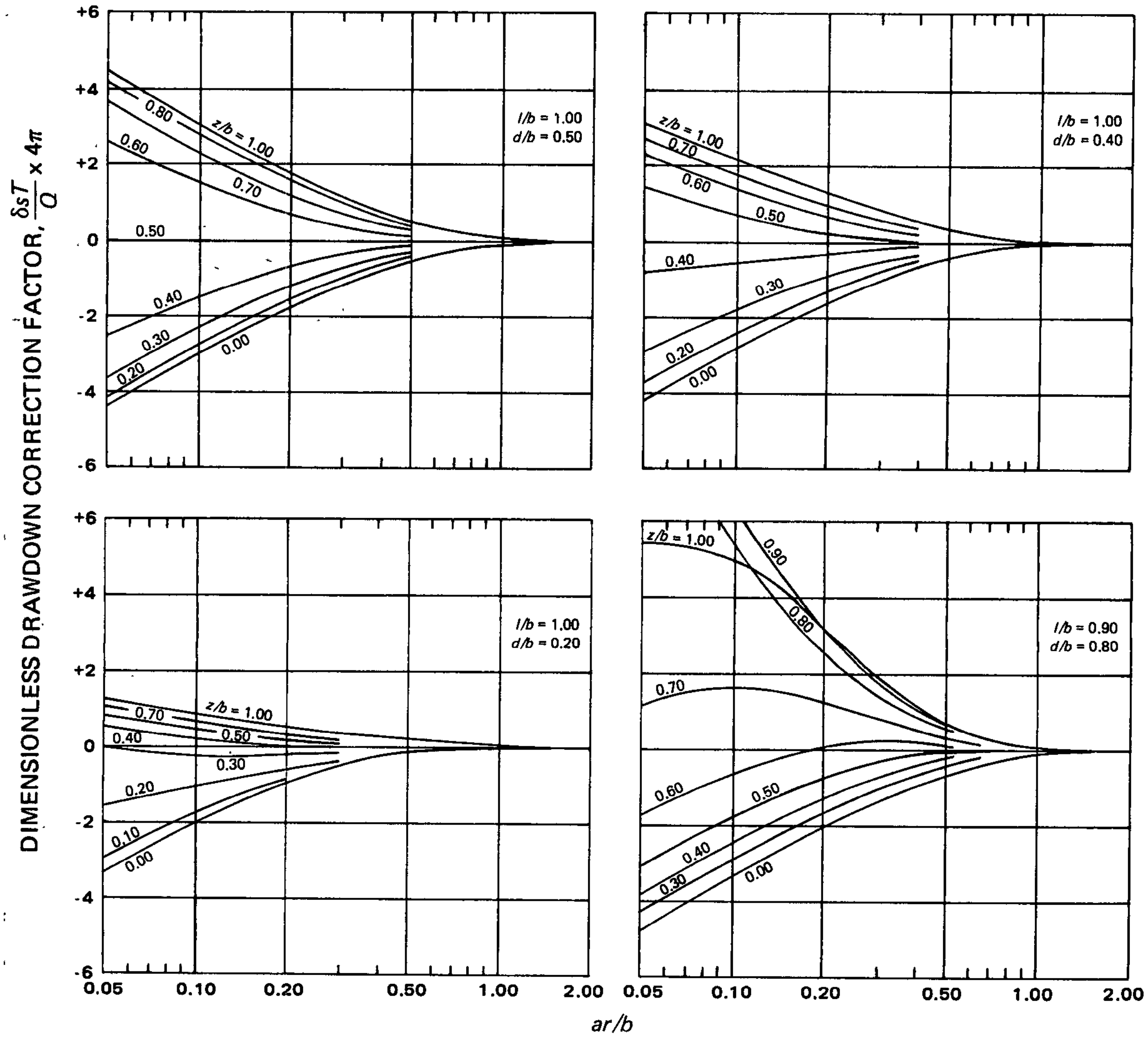

FIGURE 2.3.—Continued.

depth to top of pumped well screen (d), coded in columns 11–15, in format F5.1; the number of observation wells and (or) piezometers, coded in columns 16–20, in format I5; the smallest value of $1/u$ for which computation is desired, coded in columns 21–30, in format E10.4; the largest value of $1/u$ for which computation is desired, coded in columns 31–40, in format E10.4. The ratio of the largest $1/u$ value to the smallest $1/u$ value should be less than 10^{12}. Following this card is a group of cards containing one card for each observation well or piezometer. These cards are coded for an observation well as: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), in columns 1–5, in format F5.1; depth to bottom of observation well screen (l'), coded in columns 6–10, in format F5.1; depth to top of observation well screen (d'), coded in columns 11–15, in format F5.1. A card would be coded for a piezometer as follows: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), in columns 1–5, in format F5.1; and total depth of piezometer (z), in columns 11–15, in format F5.1. The output from this program is tables of computed function values,

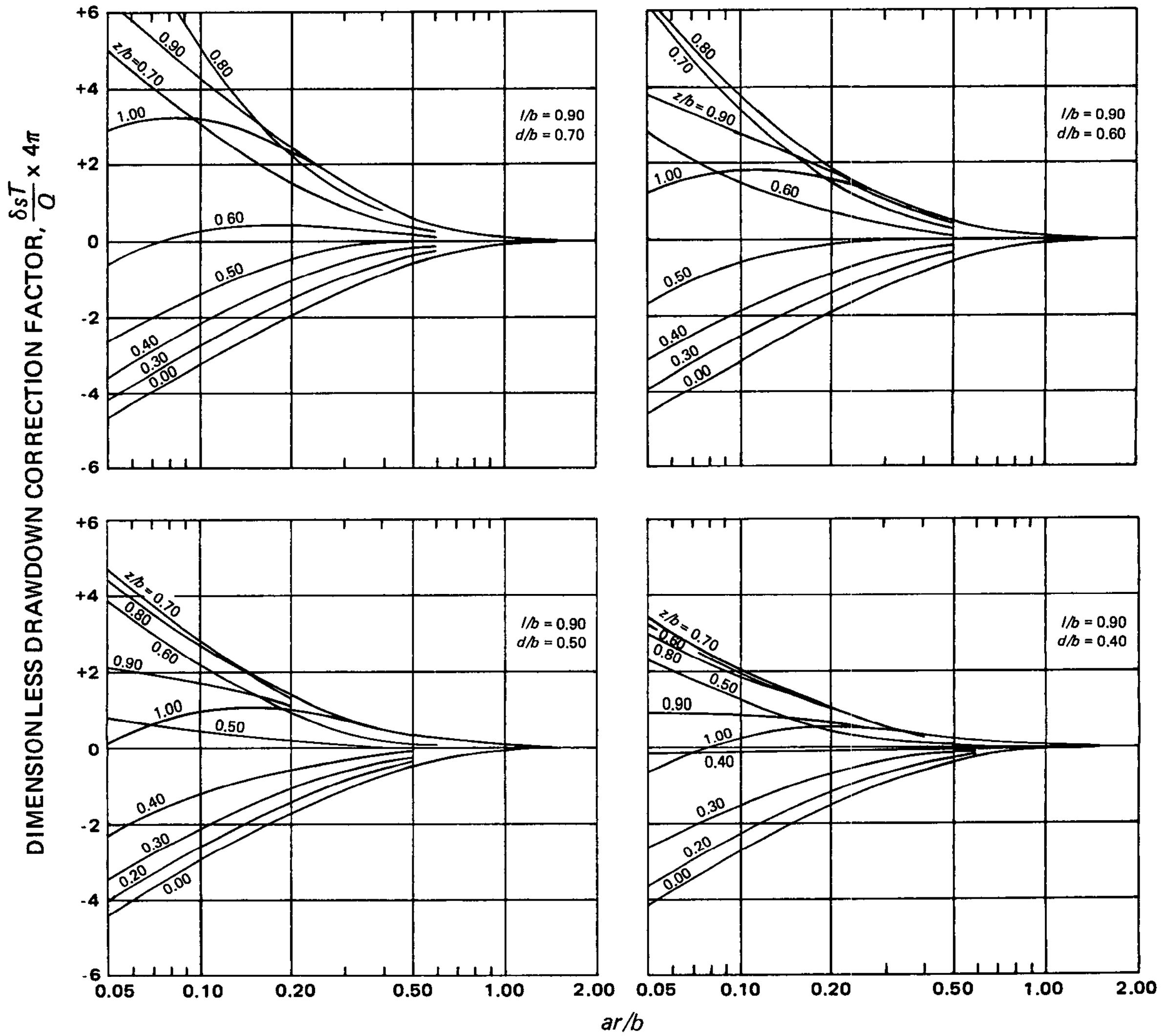

FIGURE 2.3.—Continued.

an example of which is shown in figure 2.4. Subroutines DQL12, BESK, and EXPI are from the IBM Scientific Subroutine Package and a discussion of them is in the IBM SSP manual.

Solution 3: Constant drawdown in a well in a nonleaky aquifer

Assumptions:

1. Water level in well is changed instantaneously by s_w at $t = 0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic confined aquifer.

Boundary and initial conditions:

$$s(r,0) = 0,\ r \geqslant r_w \qquad (1)$$

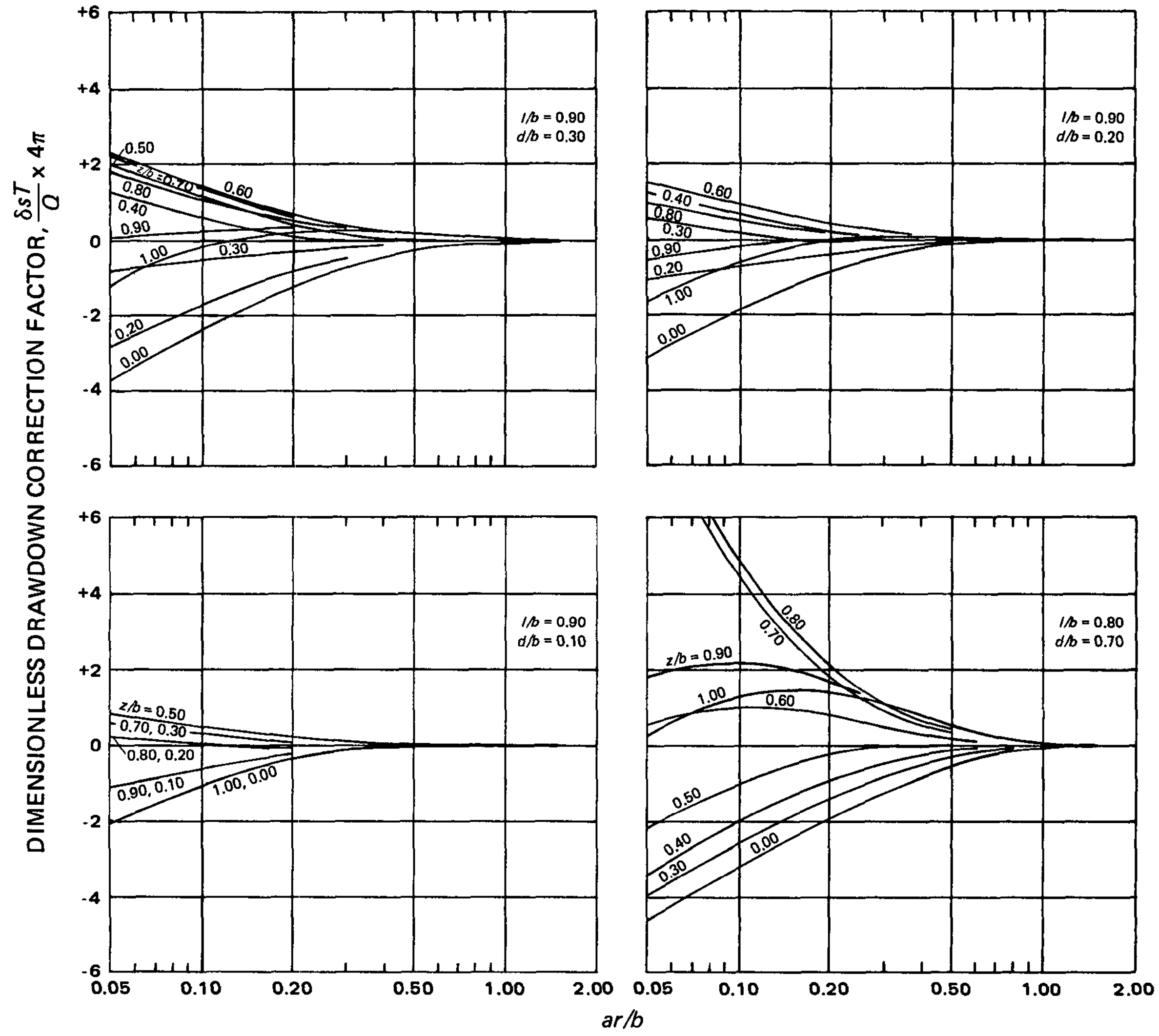

FIGURE 2.3.—Continued.

$$s(r_w,t) = \begin{cases} 0, & t < 0 \\ s_w = \text{constant}, & t \geq 0 \end{cases} \qquad (2)$$

$$s(\infty,t) = 0, \; t \geq 0 \qquad (3)$$

Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that, as the well is approached, drawdown in the aquifer approaches the constant drawdown in the well, implying no entrance loss to the well. Equation 3 states that the drawdown approaches zero as the distance from the well approaches infinity.

Solutions:

I. For the well discharge (Jacob and Lohman, 1952, p. 560):

$$Q = 2\pi T \, s_w \, G(\alpha),$$

where

$$G(\alpha) = \frac{4\alpha}{\pi} \int_0^\infty x e^{-\alpha x^2} \left\{ \frac{\pi}{2} + \tan^{-1} \left[\frac{Y_0(x)}{J_0(x)} \right] \right\} dx$$

and

$$\alpha = \frac{Tt}{Sr_w^2} .$$

II. For the drawdown in water level (Hantush, 1964a, p. 343):

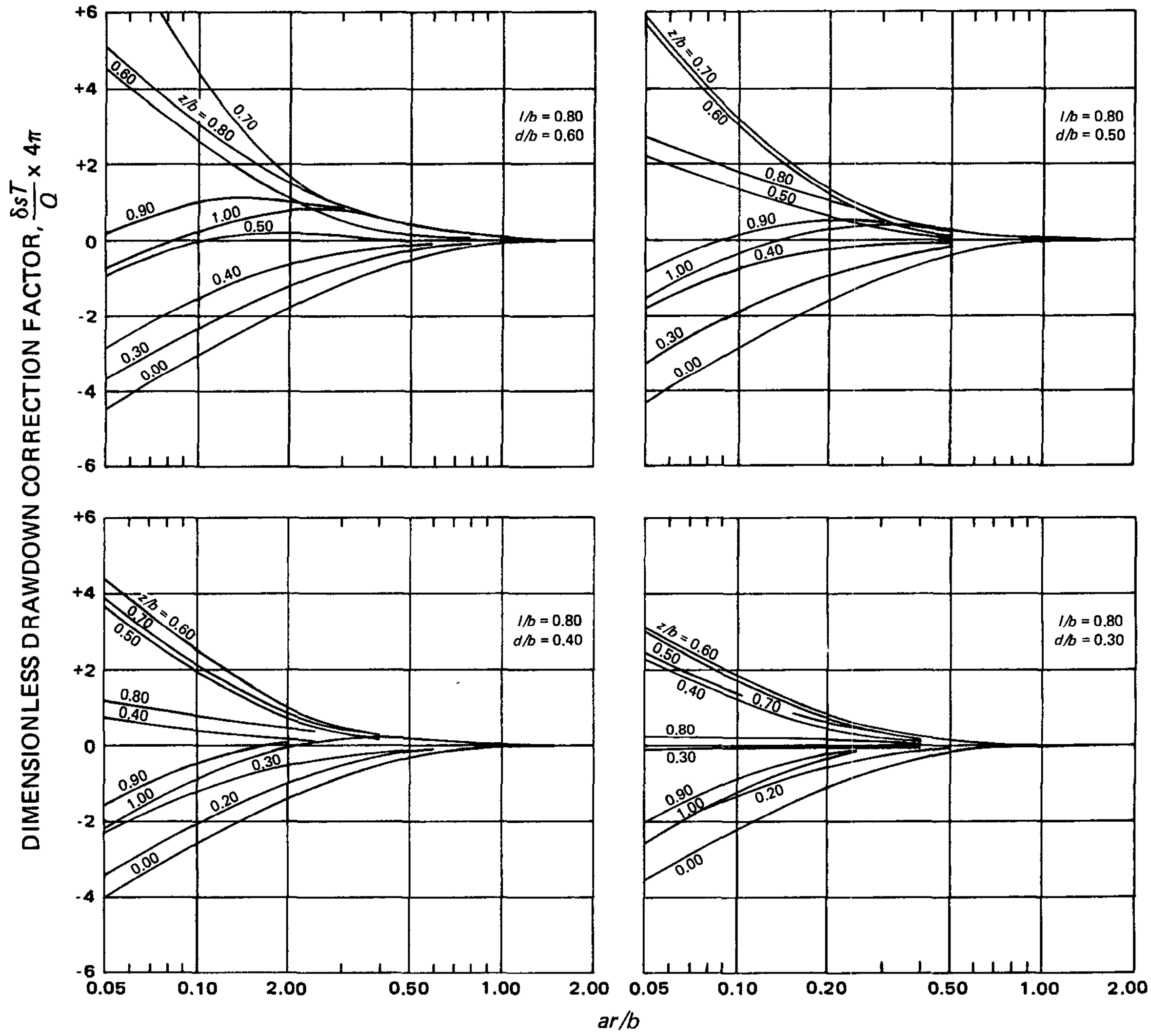

FIGURE 2.3.—Continued.

$$s = s_w A(\tau, \rho),$$

where
$$A(\tau,\rho) = 1 - \frac{2}{\pi}\int_0^\infty \frac{J_0(u)\, Y_0(\rho u) - Y_0(u) J_0(\rho u)}{J_0^2(u) + Y_0^2(u)} \exp(-\tau u^2)\frac{du}{u},$$

and
$$\tau = \alpha = \frac{Tt}{Sr_w^2},$$

$$\rho = \frac{r}{r_w}.$$

Comments:

Boundary condition 2 requires a constant drawdown in the discharging well, a condition most commonly fulfilled by a flowing well, although figure 3.1 shows the water level to be below land surface.

Figure 3.2 on plate 1 is a plot from Lohman (1972, p. 24) of dimensionless discharge ($G(\alpha)$) versus dimensionless time (α). Additional values in the range α greater than 1×10^{12} were calculated from $G(\alpha)\simeq 2/\log(2.2458\alpha)$ (Hantush, 1964a, p. 312). Function values for $G(\alpha)$ are given in table 3.1. The data curve consists of measured well discharge versus time. After the data and type curves are matched, transmissivity can be calculated from $T = Q/2\pi s_w G(\alpha)$, and the storage coefficient can be

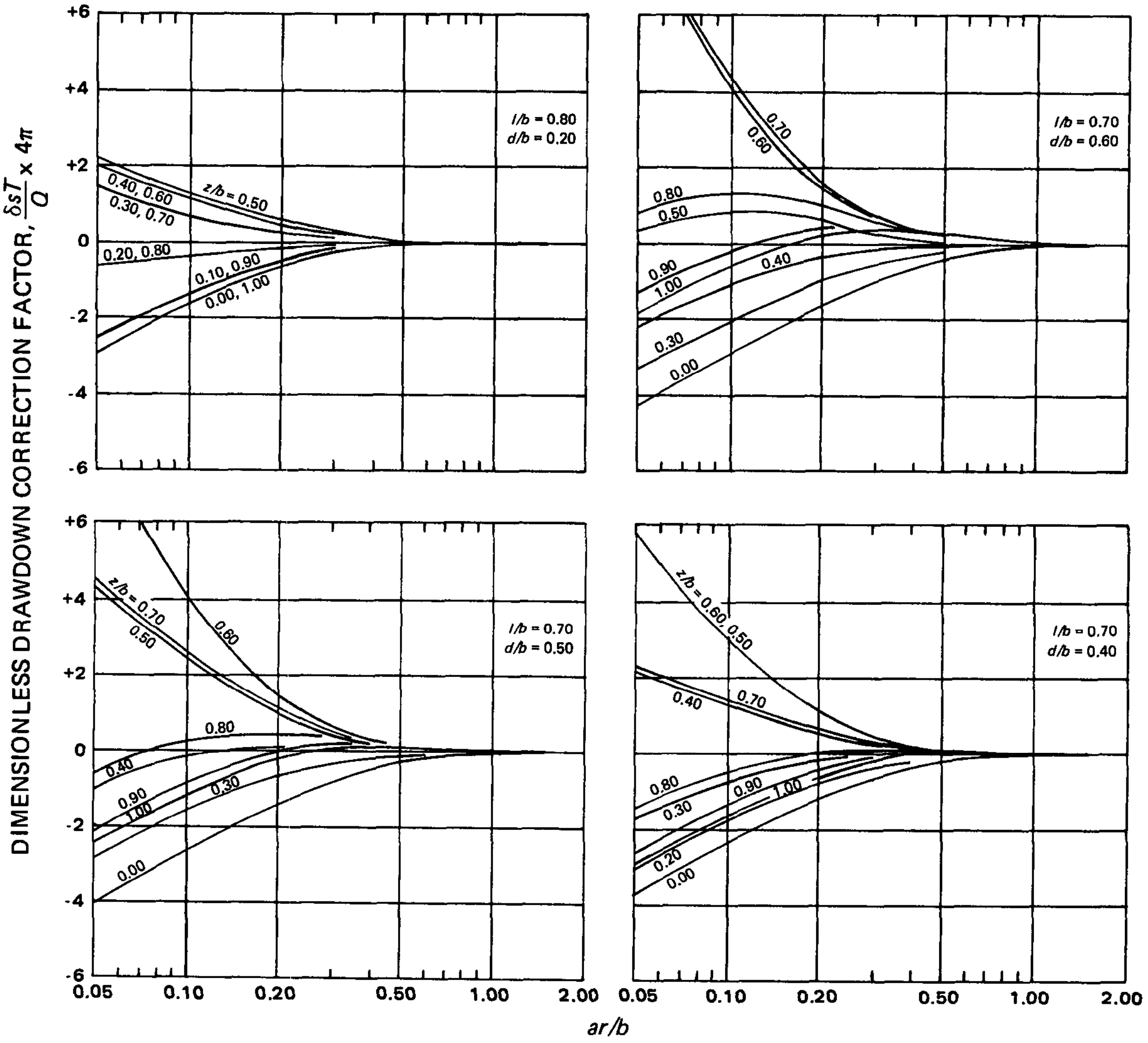

FIGURE 2.3.—Continued.

calculated from $S = Tt/\alpha r_w^2$, where $(\alpha, G(\alpha))$ and (t, Q) are matching points on the type curve and data curve, respectively.

Similarly, data curves of drawdown versus time may be matched to figure 3.3 on plate 1; this is a plot of dimensionless drawdown $(A(\tau,\rho) = s/s_w)$ versus dimensionless time $(\tau/\rho^2 = Tt/Sr^2)$. After the data and type curves are matched, the hydraulic diffusivity of the aquifer can be calculated from the equality $T/S = (\tau/\rho^2)(r^2/t)$. Usually s_w is known, and some of the uncertainty of curve matching can be eliminated by plotting s/s_w versus t because only horizontal translation is then required. If r_w is also known, the particular curve to be matched can be determined from the relation $\rho = r/r_w$. Generally, however, the effective radius, r_w, differs from the actual radius and is not known. The effective radius can often be estimated from a knowledge of the construction of the well and the water-bearing material, or it can be determined from step-drawdown tests (Rorabaugh, 1953). Figure 3.3 was plotted from table 3.2. For $\tau \leq 1 \times 10^3$, the data are from Hantush (1964a, p. 310). For $\tau > 1 \times 10^3$, values of drawdown in a leaky aquifer, as $r_w/B \to 0$, were used. (See solution 7.) Where 0.000 occurs in table 3.2, $A(\tau,\rho)$ is less than 0.0005.

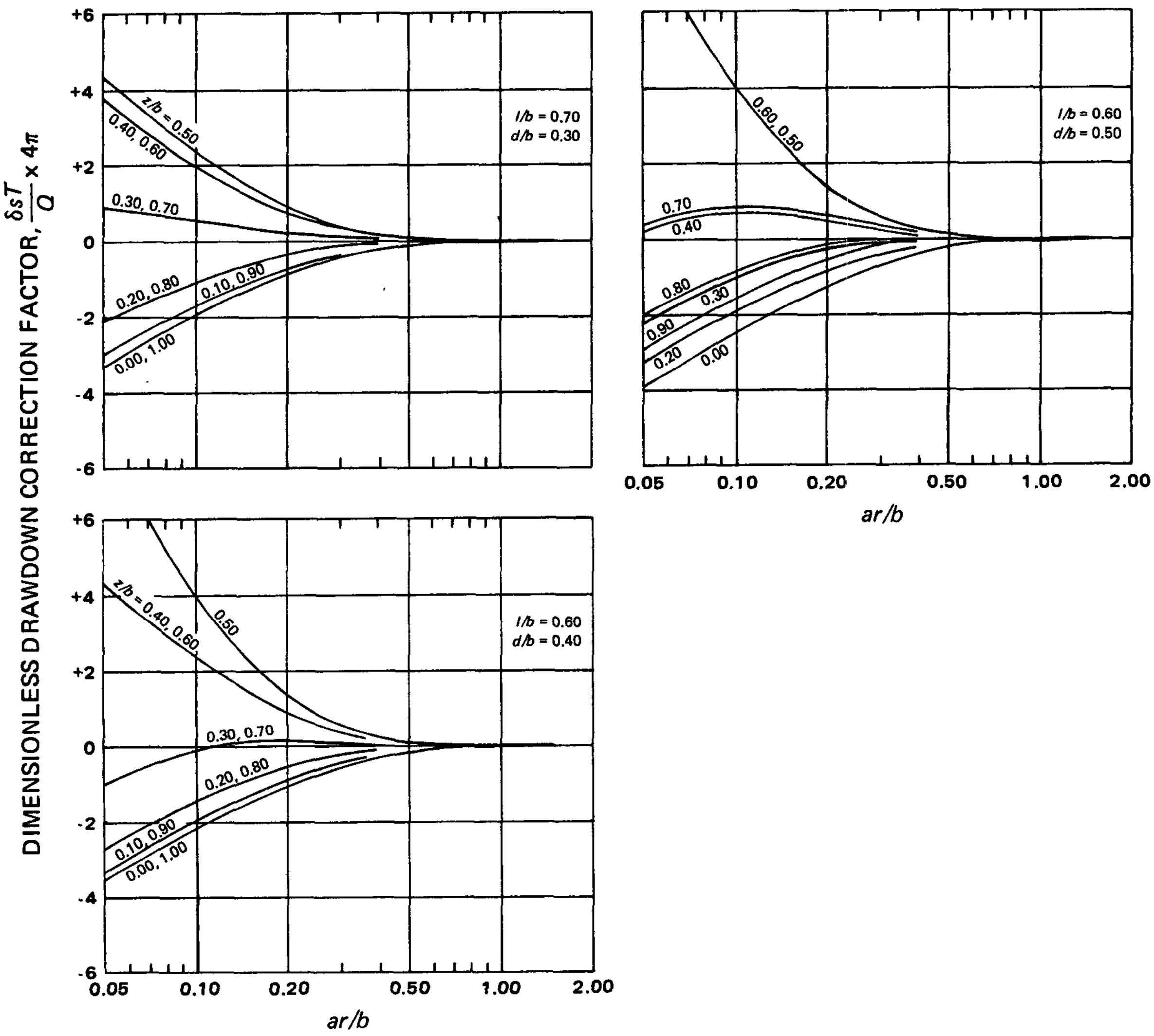

FIGURE 2.3.—Continued.

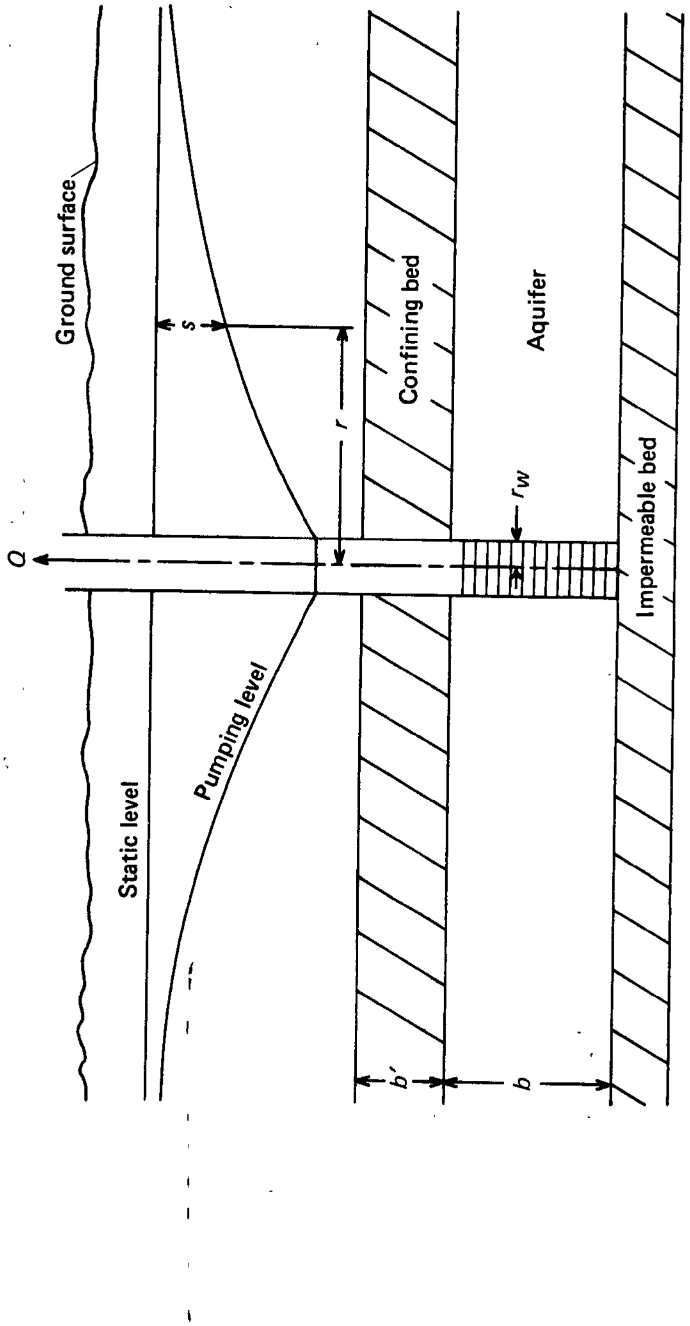

FIGURE 2.4.—Example of output from program for partial penetration in a nonleaky artesian aquifer.

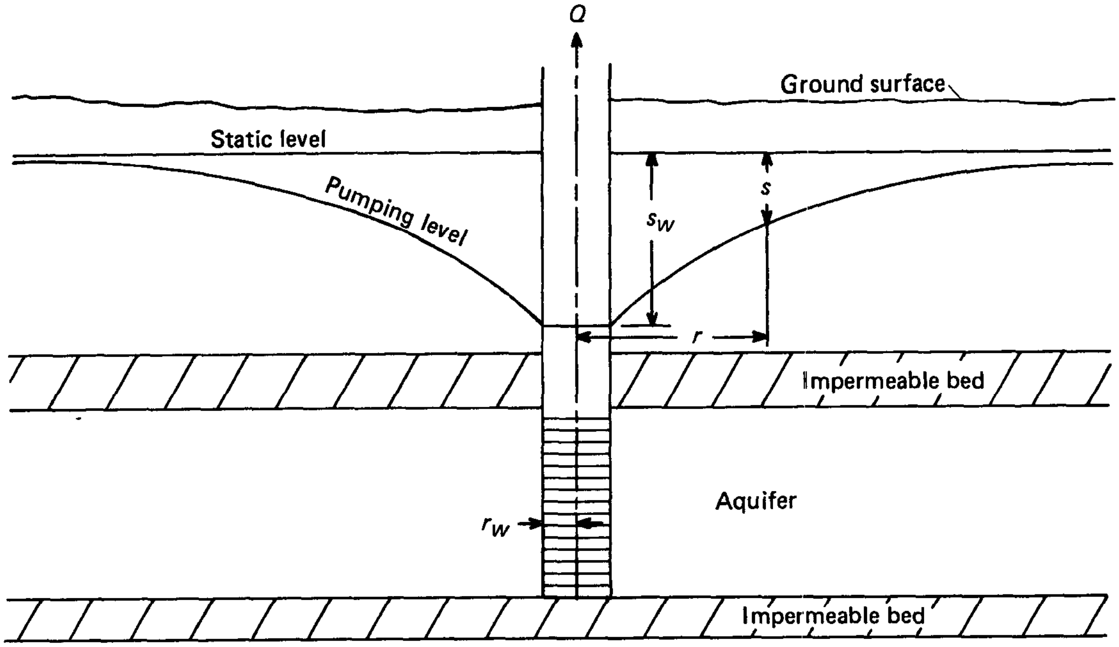

FIGURE 3.1.—Cross section through a well with constant drawdown in a nonleaky aquifer.

Solution 4: Constant discharge from a fully penetrating well in a leaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{sK'}{Tb'} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

(1)

$$s(\infty,t)=0,\ t\geq 0 \qquad (2)$$

$$Q=\begin{cases}0, & t<0\\ \text{constant}>0, & t\geq 0\end{cases} \qquad (3)$$

$$\lim_{r\to 0} r\,\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \qquad (4)$$

Equation 1 states that the initial drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equation 3 states that the discharge from the well is constant and begins at $t=0$. Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.

TABLE 3.1.—*Values of* G(α)

[Modified from Lohman (1972, p. 24)]

α	$\alpha \times 10^{-4}$	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}
1	56.9	18.34	6.13	2.249	0.985	0.534	0.346	0.251	0.1964	0.1608
2	40.4	13.11	4.47	1.716	.803	.461	.311	.232	.1841	.1524
3	33.1	10.79	3.74	1.477	.719	.427	.294	.222	.1777	.1479
4	28.7	9.41	3.30	1.333	.667	.405	.283	.215	.1733	.1449
5	25.7	8.47	3.00	1.234	.630	.389	.274	.210	.1701	.1426
6	23.5	7.77	2.78	1.160	.602	.377	.268	.206	.1675	.1408
7	21.8	7.23	2.60	1.103	.580	.367	.263	.203	.1654	.1393
8	20.4	6.79	2.46	1.057	.562	.359	.258	.200	.1636	.1380
9	19.3	6.43	2.35	1.018	.547	.352	.254	.198	.1621	.1369
α	$\alpha \times 10^{6}$	10^{7}	10^{8}	10^{9}	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}
1	0.1360	0.1177	0.1037	0.0927	0.0838	0.0764	0.0704	0.0651	0.0605	0.0566
2	.1299	.1131	.1002	.0899	.0814	.0744	.0686	.0636	.0593	.0555
3	.1266	.1106	.0982	.0883	.0801	.0733	.0677	.0628	.0586	.0549
4	.1244	.1089	.0968	.0872	.0792	.0726	.0671	.0622	.0581	
5	.1227	.1076	.0958	.0864	.0785	.0720	.0666	.0618	.0577	
6	.1213	.1066	.0950	.0857	.0779	.0716	.0662	.0615	.0574	
7	.1202	.1057	.0943	.0851	.0774	.0712	.0658	.0612	.0572	
8	.1192	.1049	.0937	.0846	.0770	.0709	.0655	.0609	.0569	
9	.1184	.1043	.0932	.0842	.0767	.0706	.0653	.0607	.0567	

TABLE 3.2.—*Values of* $A(\tau,\rho)$

[Values of $A(\tau,\rho)$ for $\tau \leq 10^3$ modified from Hantush (1964a, p 310)]

τ	ρ							
	5	10	20	50	100	200	500	1000
1 × 1	0.002							
2	.022							
3	.049							
4	.076	0.000						
5	.101	.002						
7	.142	.006						
1 × 10	.188	.016	0.000					
2	.277	.057	.001					
3	.325	.094	.004					
4	.358	.123	.009					
5	.381	.146	.016					
7	.414	.184	.031					
1 × 10^2	.446	.222	.053	0.000				
1.5	.479	.264	.085	.001				
2	.500	.291	.110	.003				
3	.528	.328	.146	.009				
5	.559	.372	.194	.026	0.000			
7	.578	.397	.223	.044	.001			
1 × 10^3	.596	.422	.254	.066	.004			
1.5	.615	.450	.287	.094	.012			
2	.627	.467	.309	.116	.021	0.000		
3	.644	.490	.338	.147	.039	.001		
5	.662	.517	.372	.186	.068	.006		
7	.673	.533	.392	.211	.089	.014		
1 × 10^4	.685	.549	.413	.237	.114	.025		
1.5	.696	.566	.435	.264	.142	.043	0.000	
2	.704	.577	.450	.283	.161	.058	.001	
3	.715	.592	.469	.308	.188	.081	.005	
5	.727	.609	.492	.337	.221	.113	.014	0.000
7	.734	.620	.506	.355	.242	.134	.025	.001
1 × 10^5	.742	.631	.520	.373	.263	.156	.039	.002
1.5	.750	.642	.532	.392	.285	.180	.058	.007
2	.755	.650	.544	.405	.300	.197	.072	.013
3	.762	.660	.558	.423	.321	.220	.094	.024
5	.771	.672	.574	.443	.345	.247	.122	.044
7	.776	.680	.584	.456	.360	.264	.141	.059
1 × 10^6	.782	.688	.594	.470	.376	.282	.160	.076
1.5	.788	.696	.604	.484	.392	.301	.181	.096
2	.792	.702	.612	.493	.403	.314	.196	.111
3	.797	.709	.622	.506	.418	.331	.216	.132
5	.803	.718	.633	.521	.436	.352	.240	.157
7	.807	.724	.641	.531	.448	.365	.255	.173
1 × 10^7	.811	.730	.648	.541	.459	.378	.270	.190
1.5	.815	.736	.656	.551	.472	.392	.287	.208
2	.818	.740	.662	.558	.480	.402	.299	.221
3	.822	.746	.669	.568	.492	.415	.314	.238
5	.827	.753	.678	.580	.506	.431	.333	.258
7	.830	.757	.684	.587	.514	.441	.344	.271
1 × 10^8	.833	.762	.690	.595	.523	.452	.357	.285
1.5	.837	.766	.696	.603	.533	.463	.370	.300
2	.839	.770	.701	.609	.540	.470	.379	.310
3	.842	.774	.706	.617	.549	.481	.391	.323
5	.846	.780	.714	.626	.560	.494	.406	.340
7	.849	.783	.718	.632	.567	.502	.415	.350
1 × 10^9	.851	.787	.723	.638	.574	.510	.425	.361
1.5	.854	.791	.728	.645	.582	.519	.435	.372
2	.856	.794	.731	.649	.587	.525	.443	.380
3	.858	.797	.736	.655	.594	.533	.452	.392
5	.861	.802	.742	.663	.603	.544	.464	.405
7	.863	.804	.746	.668	.609	.550	.472	.413
1 × 10^{10}	.865	.807	.749	.673	.615	.557	.480	.422
1.5	.867	.810	.753	.678	.621	.564	.488	.431
2	.869	.813	.756	.682	.625	.569	.494	.438
3	.871	.816	.760	.687	.631	.576	.502	.447
5	.874	.819	.765	.693	.638	.584	.512	.457
7	.875	.821	.768	.696	.643	.589	.518	.464
1 × 10^{11}	.877	.824	.770	.700	.647	.594	.524	.471
1.5								
2								
3								
5								
7								

Solution (Hantush and Jacob, 1955, p. 98):

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-z-\frac{r^2}{4B^2z}}}{z} \tag{5}$$

where $u = r^2S/4Tt$

$$B = \sqrt{\frac{Tb'}{K'}}. \tag{6}$$

Comments:

As pointed out by Hantush and Jacob (1954, p. 917), leakage is three-dimensional, but if the difference in hydraulic conductivities of the aquifer and confining bed are sufficiently great, the flow may be assumed to be vertical in the confining bed and radial in the aquifer. This relationship has been quantified by Hantush (1967, p. 587) in the condition $b/B<0.1$. In terms of relative conductivities, this would be $K/K' > 100\ b/b'$. Assumption 5, that there is no change in storage of water in the confining bed, was investigated by Neuman and Witherspoon (1969b, p. 821). They concluded that this assumption would not affect the solution if

$$\beta < 0.01, \quad \text{where } \beta = \frac{r}{4b}\sqrt{\frac{K'S_s'}{KS_s}}.$$

Assumption 4, that there is no drawdown in water level in the source bed lying above the confining bed, was also examined by Neuman and Witherspoon (1969a, p. 810). They indicated that drawdown in the source bed would have negligible effect on drawdown in the pumped aquifer for short times, that is, when $\frac{Tt}{r^2S} < 1.6\frac{\beta^2}{(r/B)^4}$. Also, they indicated (1969a, p. 811) that neglect of drawdown in the source bed is justified if $T_s > 100T$, where T_s represents the transmissivity of the source bed. Figure 4.1, a cross section through the discharging well, shows geometric relationships. Figure 4.2 on plate 1 shows plots of dimensionless drawdown compared to dimensionless time, using the notation of Cooper (1963) from Lohman (1972, pl. 3). Cooper expressed equations 5 and 6 as

$$L(u,v) = \int_u^\infty \frac{e^{-y-\frac{v^2}{y}}}{y}\,dy, \tag{7}$$

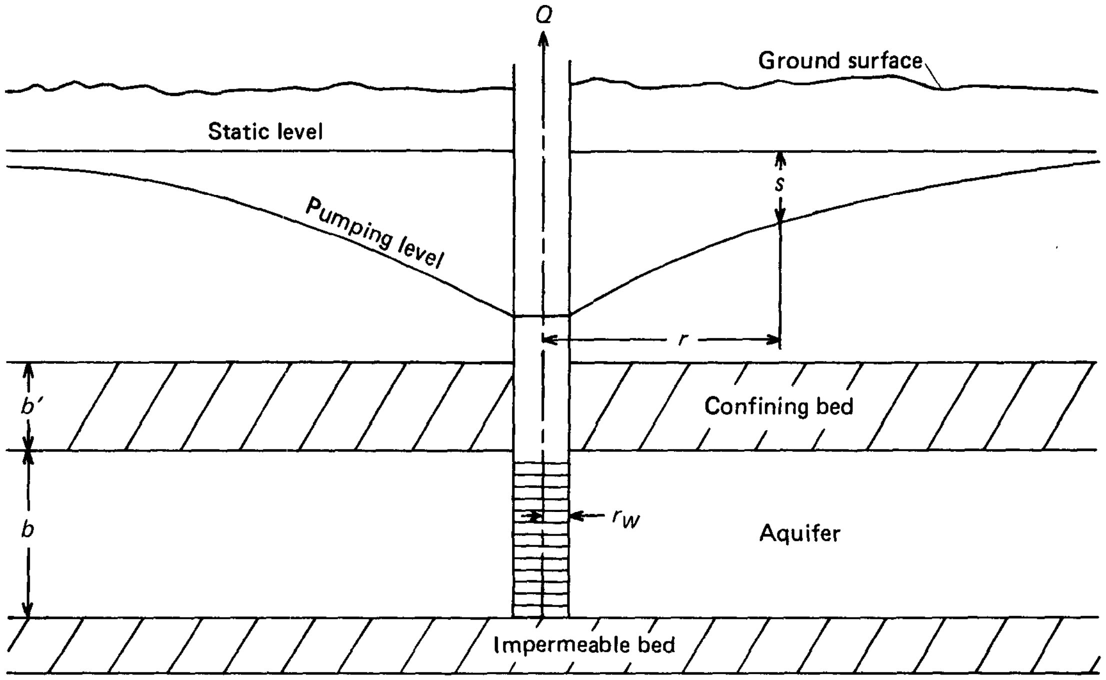

FIGURE 4.1.—Cross section through a discharging well in a leaky aquifer.

with

$$v = \frac{r}{2}\sqrt{\frac{K'}{Tb'}}. \qquad (8)$$

Cooper's type curves and equation 5 express the same function with $r/B = 2v$. Hantush (1961e) has a tabulation of equation 5, parts of which are included in table 4.1.

The observed data may be plotted in two ways (Cooper, 1963, p. C51). The measured drawdown in any one well is plotted versus t/r^2; the data are then matched to the solid-line type curves of figure 4.2. The data points are alined with the solid-line type curves either on one of them or between two of them. The parameters are then computed from the coordinates of the match points $(t/r^2, s)$ and $(1/u, L(u,v))$, and an interpolated value of v from the equations

$$T = \frac{Q}{4\pi}\frac{L(u,v)}{s}, \qquad (9)$$

$$S = 4T\frac{t/r^2}{1/u}, \qquad (10)$$

and

$$\frac{K'}{b'} = 4T\frac{v^2}{r^2}.$$

Drawdown measured at the same time but in different observation wells at different distances can be plotted versus t/r^2 and matched to the dashed-line type curves of figure 4.2. The data are matched so as to aline with the dashed-line curves, either on one or between two of them. From the match-point coordinates $(s, t/r^2)$ and $(L(u,v), 1/u)$ and an interpolated value of v^2/u, T and S are computed from equations 9 and 10 and the remaining parameter from

$$K'/b' = S\frac{v^2/u}{t}.$$

The region $v^2/u \geq 8$ and $L(u,v) \geq 10^{-2}$ corresponds to steady-state conditions.

TABLE 4.1.—*Selected values of* W(u,r/B)

[From Hantush (1961e)]

u	r/B							
	0.001	0.003	0.01	0.03	0.1	0.3	1	3
1×10^{-6}	13.0031	11.8153	9.4425	7.2471	4.8541	2.7449	0.8420	0.0695
2	12.4240	11.6716						
3	12.0581	11.5098	9.4425					
5	11.5795	11.2248	9.4413					
7	11.2570	10.9951	9.4361					
1×10^{-5}	10.9109	10.7228	9.4176					
2	10.2301	10.1332	9.2961	7.2471				
3	9.8288	9.7635	9.1499	7.2470				
5	9.3213	9.2818	8.8827	7.2450				
7	8.9863	8.9580	8.6625	7.2371				
1×10^{-4}	8.6308	8.6109	8.3983	7.2122				
2	7.9390	7.9290	7.8192	7.0685				
3	7.5340	7.5274	7.4534	6.9068	4.8541			
5	7.0237	7.0197	6.9750	6.6219	4.8530			
7	6.6876	6.6848	6.6527	6.3923	4.8478			
1×10^{-3}	6.3313	6.3293	6.3069	6.1202	4.8292			
2	5.6393	5.6383	5.6271	5.5314	4.7079	2.7449		
3	5.2348	5.2342	5.2267	5.1627	4.5622	2.7448		
5	4.7260	4.7256	4.7212	4.6829	4.2960	2.7428		
7	4.3916	4.3913	4.3882	4.3609	4.0771	2.7350		
1×10^{-2}	4.0379	4.0377	4.0356	4.0167	3.8150	2.7104		
2	3.3547	3.3546	3.3536	3.3444	3.2442	2.5688		
3	2.9591	2.9590	2.9584	2.9523	2.8873	2.4110	.8420	
5	2.4679	2.4679	2.4675	2.4642	2.4271	2.1371	.8409	
7	2.1508	2.1508	2.1506	2.1483	2.1232	1.9206	.8360	
1×10^{-1}	1.8229	1.8229	1.8227	1.8213	1.8050	1.6704	.8190	
2	1.2226	1.2226	1.2226	1.2220	1.2155	1.1602	.7148	.0695
3	.9057	.9057	.9056	.9053	.9018	.8713	.6010	.0694
5	.5598	.5598	.5598	.5596	.5581	.5453	.4210	.0681
7	.3738	.3738	.3738	.3737	.3729	.3663	.2996	.0639
1×10^{0}	.2194	.2194	.2194	.2193	.2190	.2161	.1855	.0534
2	.0489	.0489	.0489	.0489	.0488	.0485	.0444	.0210
3	.0130	.0130	.0130	.0130	.0130	.0130	.0122	.0071
5	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0008
7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

The drawdown in the steady-state region is given by the equation (Jacob, 1946, eq. 15)

$$s = \frac{Q}{2\pi T} K_0(x),$$

where $K_0(x)$ is the zero-order modified Bessel function of the second kind and

$$x = r\sqrt{\frac{K'}{Tb'}}.$$

Data for steady-state conditions can be analyzed using figure 4.3 on plate 1. The drawdowns are plotted versus r and matched to figure 4.3. After choosing a convenient match point with coordinates (s,r) and $(K_0(x),x)$ the parameters are computed from the equations

$$T = \frac{Q}{2\pi s} K_0(x) \text{ and } \frac{K'}{b'} = \frac{xT}{r^2}.$$

Values of $K_0(x)$ from Hantush (1956) are given in table 4.2.

A FORTRAN program for generating type-curve function values of equation 7 is listed in table 4.3. Using the notation $L(u,v)$ of Cooper (1963), the function is evaluated as follows. For $u \geqslant 1$,

$$L(u,v) = \int_u^\infty (1/y)\exp(-y-v^2/y)\,dy = \int_u^\infty f(y)\,dy.$$

This integral is transformed into the form

$$\int_0^\infty e^{-x}\left[\exp\left(-u-\frac{v^2}{x+u}\right)\frac{1}{x+u}\right]dx$$

evaluated by a Gaussian-Laguerre quadrature formula. For $v^2<u<1$,

$$L(u,v) = \int_1^\infty f(y)\,dy + \int_u^1 f(y)\,dy.$$

The first integral is evaluated by a Gaussian-Laguerre quadrature formula, as previously described. The second integral is evaluated using a series expansion, as

$$\int_u^1 f(y)dy = s(1)-s(u),$$

where

$$s = \log u\left[\sum_{n=0}^{\infty}\frac{(v^2)^n}{(n!)^2}\right] + \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{m}\left[u^m-\left(\frac{v^2}{u}\right)^m\right]\left[\sum_{n=0}^{\infty}\frac{(v^2)^n}{(m+n)!n!}\right]\right].$$

For $u < 1$ and $u \leqslant v^2$,

$$L(u,v) = 2K_0(2v) - \int_{\frac{v^2}{u}}^{\infty} f(y)\,dy$$

(Cooper, 1963, p. C50),

where K_0 is the zero-order modified Bessel function of the second kind. The integral in the above expression is evaluated by the Gaussian-Laguerre procedure, as described previously.

TABLE 4.2.—*Selected values of $K_o(x)$*

[From Hantush (1956, p. 704)]

N	$x=NX10^{-2}$	$x=NX10^{-1}$	$x=N$
1	4.7212	2.4271	0.4210
1.5	4.3159	2.0300	.2138
2	4.0285	1.7527	.1139
3	3.6235	1.3725	.0347
4	3.3365	1.1145	.0112
5	3.1142	.9244	.0037
6	2.9329	.7775	------
7	2.7798	.6605	------
8	2.6475	.5653	------
9	2.5310	.4867	------

Input data for this program consist of three cards with the numeric data coded by specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of $1/u$ for which computation is desired, coded in columns 1–10 in format E10.5; the largest value of $1/u$ for which computation is desired, coded in columns 11–20 in format E10.5. The table will include a range of $1/u$ values spanning these two coded values if the span is less than or equal to 12 log cycles. The next two cards contain 12 values of r/B, all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Zero (or blank) coding is permissible in this field, but computation will terminate with the first zero (or blank) value encountered. An example of the output from this program is shown in figure 4.4.

```
W(U,R/B)

          | R/B
   1/U    |  0.10E-05  0.30E-05  0.10E-04  0.30E-04  0.10E-03  0.30E-03  0.10E-02  0.30E-02  0.10E-01
0.100E 01    0.2194    0.2194    0.2194    0.2194    0.2194    0.2194    0.2194    0.2194    0.2194
0.150E 01    0.3984    0.3984    0.3984    0.3984    0.3984    0.3984    0.3984    0.3984    0.3984
0.200E 01    0.5598    0.5598    0.5598    0.5598    0.5598    0.5598    0.5598    0.5598    0.5598
0.300E 01    0.8289    0.8289    0.8289    0.8289    0.8289    0.8289    0.8289    0.8289    0.8289
0.500E 01    1.2226    1.2226    1.2226    1.2226    1.2226    1.2226    1.2226    1.2226    1.2226
0.700E 01    1.5066    1.5066    1.5066    1.5066    1.5066    1.5066    1.5066    1.5066    1.5065
0.100E 02    1.8229    1.8229    1.8229    1.8229    1.8229    1.8229    1.8229    1.8229    1.8227
0.150E 02    2.1964    2.1964    2.1964    2.1964    2.1964    2.1964    2.1964    2.1964    2.1961
0.200E 02    2.4679    2.4679    2.4679    2.4679    2.4679    2.4679    2.4679    2.4679    2.4675
0.300E 02    2.8570    2.8570    2.8570    2.8570    2.8570    2.8570    2.8570    2.8570    2.8564
0.500E 02    3.3547    3.3547    3.3547    3.3547    3.3547    3.3547    3.3547    3.3546    3.3536
0.700E 02    3.6855    3.6855    3.6855    3.6855    3.6855    3.6855    3.6855    3.6854    3.6839
0.100E 03    4.0379    4.0379    4.0379    4.0379    4.0379    4.0379    4.0379    4.0377    4.0356
0.150E 03    4.4401    4.4401    4.4401    4.4401    4.4401    4.4401    4.4400    4.4397    4.4365
0.200E 03    4.7261    4.7261    4.7261    4.7261    4.7261    4.7261    4.7260    4.7257    4.7212
0.300E 03    5.1299    5.1299    5.1299    5.1299    5.1299    5.1299    5.1298    5.1292    5.1226
0.500E 03    5.6394    5.6394    5.6394    5.6394    5.6394    5.6394    5.6393    5.6383    5.6271
0.700E 03    5.9753    5.9753    5.9753    5.9753    5.9753    5.9753    5.9751    5.9737    5.9580
0.100E 04    6.3315    6.3315    6.3315    6.3315    6.3315    6.3315    6.3313    6.3293    6.3069
0.150E 04    6.7367    6.7367    6.7367    6.7367    6.7367    6.7366    6.7363    6.7333    6.6997
0.200E 04    7.0242    7.0242    7.0242    7.0242    7.0242    7.0241    7.0237    7.0197    6.9750
0.300E 04    7.4295    7.4295    7.4295    7.4295    7.4295    7.4294    7.4287    7.4228    7.3561
0.500E 04    7.9402    7.9402    7.9402    7.9402    7.9402    7.9401    7.9389    7.9290    7.8192
0.700E 04    8.2766    8.2766    8.2766    8.2766    8.2766    8.2764    8.2748    8.2609    8.1092
0.100E 05    8.6332    8.6332    8.6332    8.6332    8.6332    8.6330    8.6307    8.6109    8.3983
```

FIGURE 4.4.—Example of output from program for computing drawdown due to constant discharge from a well in a leaky artesian aquifer.

Solution 5: Constant discharge from a well in a leaky aquifer with storage of water in the confining beds

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain and underlain everywhere by confining beds having hydraulic conductivities K' and K'', thicknesses b' and b'', and storage coefficients S' and S'', respectively, which are constant in space and time.
4. Flow in the aquifer is two dimensional and radial in the horizontal plane and flow in confining beds is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining beds.
5. Conditions at the far surfaces of the confining beds are (fig. 5.1):
   - Case 1. Constant-head plane sources above and below.
   - Case 2. Impermeable beds above and below.
   - Case 3. Constant-head plane source above and impermeable bed below.

Differential equations:

For the upper confining bed

$$\frac{\partial^2 s_1}{\partial z^2} = \frac{S'}{K'b'} \frac{\partial s_1}{\partial t} \qquad (1)$$

For the aquifer

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K'}{T}\frac{\partial}{\partial z} s_1(r,b',t) - \frac{K''}{T}\frac{\partial}{\partial z} s_2(r,b'+b,\, t) = \frac{S}{T}\frac{\partial s}{\partial t} \qquad (2)$$

For the lower confining bed

$$\frac{\partial^2 s_2}{\partial z^2} = \frac{S''}{K''b''} \frac{\partial s_2}{\partial t} \qquad (3)$$

Equations 1 and 3 are, respectively, the differential equations for nonsteady vertical flow in the upper and lower semipervious beds. Equation 2 is the differential equation for nonsteady two-dimensional radial flow in an aquifer with leakage at its upper and lower boundaries.

Boundary and initial conditions:

Case 1: For the upper confining bed

$$s_1(r,z,0)=0 \qquad (4)$$

$$s_1(r,0,t)=0 \qquad (5)$$

$$s_1(r,b',t)=s(r,t) \qquad (6)$$

For the aquifer

$$s(r,0)=0 \qquad (7)$$

$$s(\infty,t)=0 \qquad (8)$$

$$\lim_{r \to 0} r \frac{\partial s(r,t)}{\partial r} = -\frac{Q}{2\pi T} \qquad (9)$$

For the lower confining bed

$$s_2(r,z,0)=0 \qquad (10)$$

$$s_2(r,b'+b+b'',t)=0 \qquad (11)$$

$$s_2(r,b'+b,t)=s(r,t) \qquad (12)$$

Case 2: Same as case 1, with conditions 5 and 11 being replaced, respectively, by

$$\frac{\partial s_1(r,0,t)}{\partial z} = 0 \qquad (13)$$

$$\frac{\partial s_2(r,b'+b+b'')}{\partial z} = 0 \qquad (14)$$

Case 3: Same as case 1, with condition 11 being replaced by condition 14.

Equations 4, 7, and 10 state that initially the drawdown is zero in the aquifer and within each confining bed. Equation 5 states that a plane of zero drawdown occurs at the top of the upper confining bed. Equations 6 and 12 state that, at the upper and lower boundaries of the aquifer, drawdown in the aquifer is equal to drawdown in the confining beds. Equation 8 states that drawdown is small at a large distance from the pumping well. Equation 9 states that, near the pumping well, the flow is equal to the discharge rate. Equation 11 states that a plane of zero drawdown is at the base of the lower confining bed. Equation 13 states that there is no flow across the top of the upper confining bed. Equation 14 states that no flow occurs across the base of the lower confining bed.

Solutions (Hantush, 1960, p. 3716):

I. For small values of time (t less than both $b'S'/10K'$ and $b''S''/10K''$):

$$s = \frac{Q}{4\pi T} H(u,\beta) , \qquad (15)$$

where
$$u = \frac{r^2S}{4Tt}$$

and
$$\beta = \frac{r}{4}\left(\sqrt{\frac{K'S'}{b'TS}} + \sqrt{\frac{K''S''}{b''TS}}\right)$$

$$H(u,\beta) = \int_u^\infty \frac{e^{-y}}{y}\, erfc \frac{\beta\sqrt{u}}{\sqrt{y(y-u)}}\, dy$$

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}\, dy .$$

II. For large values of time:

A. Case 1, t greater than both $5b'S'/K'$ and $5b''S''/K''$

$$s = \frac{Q}{4\pi T} W(u\delta_1,\alpha) , \qquad (16)$$

where u is as defined previously

and
$$\delta_1 = 1 + (S' + S'')/3S,$$

$$\alpha = r\sqrt{\frac{K'/b'}{T} + \frac{K''/b''}{T}}$$

$$W(u,x) = \int_u^\infty \frac{\exp\,(-y - x^2/4y)}{y}\, dy .$$

B. Case 2, t greater than both $10b'S'/K'$ and $10b''S''/K''$

$$s = \frac{Q}{4\pi T} W(u\delta_2) , \qquad (17)$$

where
$$\delta_2 = 1 + (S' + S'')/S$$

$$W(u) = \int_u^\infty \frac{e^{-y}}{y}\, dy .$$

C. Case 3, t greater than both $5b'S'/K'$ and $10b''S''/K''$

$$s = \frac{Q}{4\pi T} W\left(u\delta_3,\, r\sqrt{\frac{K'/b'}{T}}\right) , \qquad (18)$$

where

$$\delta_3 = 1 + (S'' + S'/3)/S$$

and $W(u,x)$ is as defined in case 1.

Comments:

A cross section through the discharging well is shown in figure 5.1. The flow system is actually three-dimensional in such a geometric configuration. However, as stated by Hantush (1960, p. 3713), if the hydraulic conductivity in the aquifer is sufficiently greater than the hydraulic conductivity of the confining beds, flow will be approximately radial in the aquifer and approximately vertical in the confining beds. A complete solution to this flow problem has not been published. Neuman and Witherspoon (1971, p. 250, eq. II–161) developed a complete solution for case 1 but did not tabulate it. Hantush's solutions, which have been tabulated, are solutions that are applicable for small and large values of time but not for intermediate times.

The "early" data (data collected for small values of t) can be analyzed using equation 15. Figure 5.2 on plate 1 shows plots of $H(u,\beta)$ from Lohman (1972, pl. 4). Hantush (1961d) has an extensive tabulation of $H(u,\beta)$, a part of which is given in table 5.1. The corresponding data curves would consist of observed drawdown versus t/r^2. Superposing the data curves on the type curves and matching the two, with graph axes parallel, so that the data curves lie on or between members of the type-curve family and choosing a convenient match point ($H(u,\beta)$, $1/u$), T and S are computed by

$$T = \frac{Q}{4\pi s} H(u,\beta) ,$$

$$S = 4T\frac{t}{r^2}\bigg/\frac{1}{u} .$$

If simplifying conditions are applicable, it is possible to compute the product $K'S'$ from the β value. If $K''S''=0$, $K'S'=16\beta^2b'TS/r^2$, and if $K''S''=K'S'$,

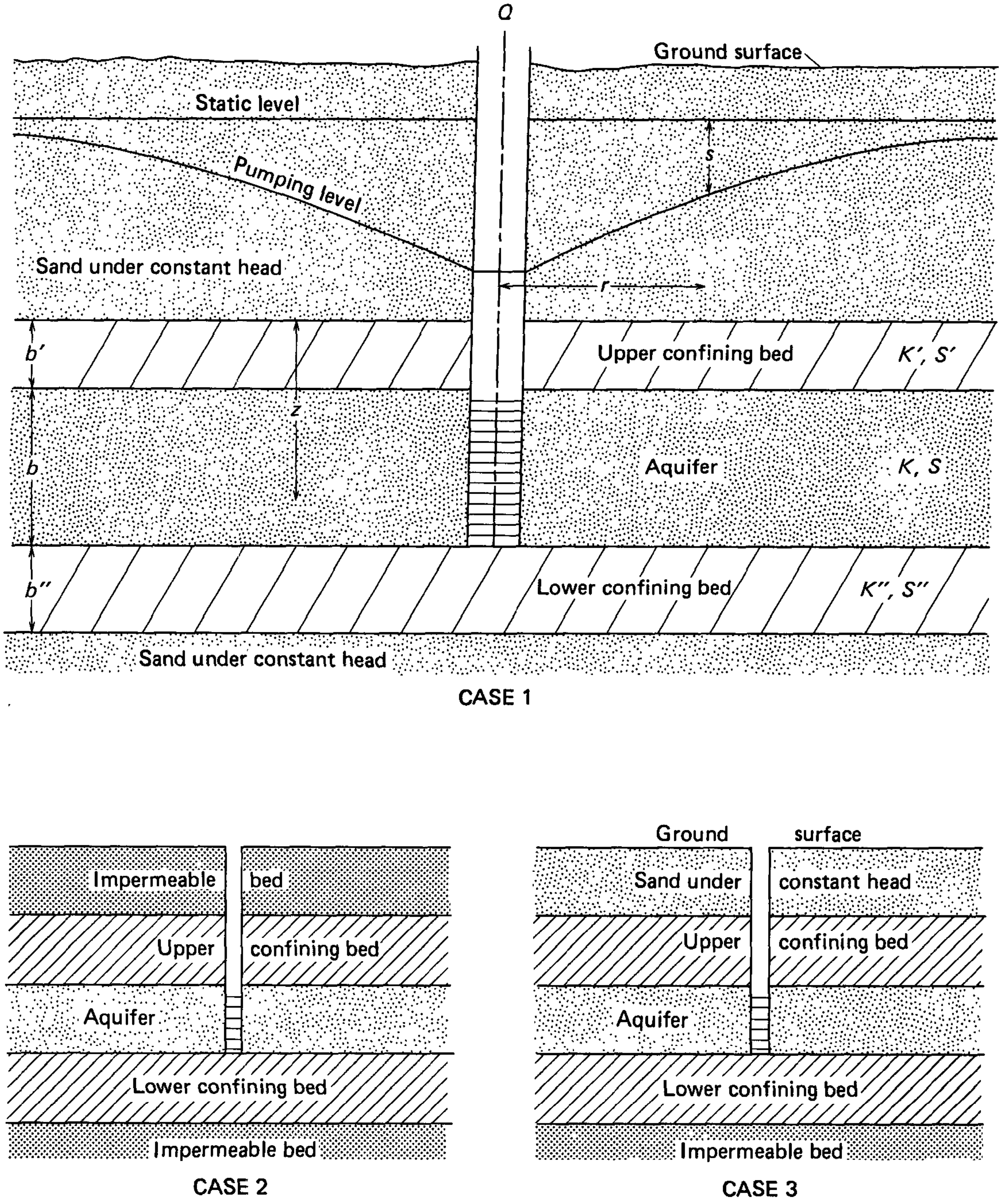

FIGURE 5.1.—Cross sections through discharging wells in leaky aquifers with storage of water in the confining beds, illustrating three different cases of boundary conditions.

$$K'S' = \frac{16\beta^2}{r^2}\,TS\,\frac{b'b''}{b'+b''+2\sqrt{b'b''}} \; .$$

The curves in figure 5.2 are very similar from $\beta=0$ to about $\beta=0.5$. Therefore, the β values in this range are indeterminate. There is also uncertainty in curve matching for all β values because of the fact that it is a family of curves whose shapes change gradually with β. This uncertainty will be increased if the data covers a small range of t values. The problem

TABLE 5.1.—*Values of* $H(u,\beta)$ *for selected values of* u *and* β

[From Hantush (1961d) Numbers in parentheses are powers of 10 by which the other numbers are multiplied, for example 963(−4) = 0.0963]

u	β							
	0 03	0 1	0.3	1	3	10	30	100
1×10^{-9}	12.3088	11.1051	10.0066	8.8030	7.7051	6.5033	5.4101	4.2221
2	11.9622	10.7585	9.6602	8.4566	7.3590	6.1579	5.0666	3.8839
3	11.7593	10.5558	9.4575	8.2540	7.1565	5.9561	4.8661	3.6874
5	11.5038	10.3003	9.2021	7.9987	6.9016	5.7020	4.6142	3.4413
7	11.3354	10.1321	9.0339	7.8306	6.7337	5.5348	4.4487	3.2804
1×10^{-8}	11.1569	9.9538	8.8556	7.6525	6.5558	5.3578	4.2737	3.1110
2	10.8100	9.6071	8.5091	7.3063	6.2104	5.0145	3.9352	2.7858
3	10.6070	9.4044	8.3065	7.1039	6.0085	4.8141	3.7383	2.5985
5	10.3511	9.1489	8.0512	6.8490	5.7544	4.5623	3.4919	2.3662
7	10.1825	8.9806	7.8830	6.6811	5.5872	4.3969	3.3307	2.2159
1×10^{-7}	10.0037	8.8021	7.7048	6.5032	5.4101	4.2221	3.1609	2.0591
2	9.6560	8.4554	7.3585	6.1578	5.0666	3.8839	2.8348	1.7633
3	9.4524	8.2525	7.1560	5.9559	4.8661	3.6874	2.6469	1.5966
5	9.1955	7.9968	6.9009	5.7018	4.6141	3.4413	2.4137	1.3944
7	9.0261	7.8283	6.7329	5.5346	4.4486	3.2804	2.2627	1.2666
1×10^{-6}	8.8463	7.6497	6.5549	5.3575	4.2736	3.1110	2.1051	1.1361
2	8.4960	7.3024	6.2091	5.0141	3.9350	2.7857	1.8074	.8995
3	8.2904	7.0991	6.0069	4.8136	3.7382	2.5984	1.6395	.7725
5	8.0304	6.8427	5.7523	4.5617	3.4917	2.3661	1.4354	.6256
7	7.8584	6.6737	5.5847	4.3962	3.3304	2.2158	1.3061	.5375
1×10^{-5}	7.6754	6.4944	5.4071	4.2212	3.1606	2.0590	1.1741	.4519
2	7.3170	6.1453	5.0624	3.8827	2.8344	1.7632	.9339	.3091
3	7.1051	5.9406	4.8610	3.6858	2.6464	1.5965	.8046	.2402
5	6.8353	5.6821	4.6075	3.4394	2.4131	1.3943	.6546	.1685
7	6.6553	5.5113	4.4408	3.2781	2.2619	1.2664	.5643	.1300
1×10^{-4}	6.4623	5.3297	4.2643	3.1082	2.1042	1.1359	.4763	963(−4)
2	6.0787	4.9747	3.9220	2.7819	1.8062	.8992	.3287	494(−4)
3	5.8479	4.7655	3.7222	2.5937	1.6380	.7721	.2570	315(−4)
5	5.5488	4.4996	3.4711	2.3601	1.4335	.6252	.1818	166(−4)
7	5.3458	4.3228	3.3062	2.2087	1.3039	.5370	.1412	103(−4)
1×10^{-3}	5.1247	4.1337	3.1317	2.0506	1.1715	.4513	.1055	390(−5)
2	4.6753	3.7598	2.7938	1.7516	.9305	.3084	551(−4)	169(−5)
3	4.3993	3.5363	2.5969	1.5825	.8006	.2394	355(−4)	713(−6)
5	4.0369	3.2483	2.3499	1.3767	.6498	.1677	190(−4)	205(−6)
7	3.7893	3.0542	2.1877	1.2460	.5589	.1292	120(−4)	821(−7)
1×10^{-2}	3.5195	2.8443	2.0164	1.1122	.4702	955(−4)	695(−5)	274(−7)
2	2.9759	2.4227	1.6853	.8677	.3214	487(−4)	205(−5)	226(−8)
3	2.6487	2.1680	1.4932	.7353	.2491	308(−4)	888(−6)	
5	2.2312	1.8401	1.2535	.5812	.1733	160(−4)	261(−6)	
7	1.9558	1.6213	1.0979	.4880	.1325	982(−5)	106(−6)	
1×10^{-1}	1.6667	1.3893	.9358	.3970	966(−4)	552(−5)	365(−7)	
2	1.1278	.9497	.6352	.2452	468(−4)	149(−5)	307(−8)	
3	.8389	.7103	.4740	.1729	281(−4)	592(−6)		
5	.5207	.4436	.2956	.1006	130(−4)	151(−6)		
7	.3485	.2980	.1985	646(−4)	714(−5)	534(−7)		
1×1	.2050	.1758	.1172	365(−4)	337(−5)	151(−7)		
2	458(−4)	395(−4)	264(−4)	760(−5)	487(−6)			
3	122(−4)	106(−4)	707(−5)	196(−5)	102(−6)			
5	108(−5)	934(−6)	624(−6)	167(−6)	672(−8)			
7	109(−6)	941(−7)	629(−7)	165(−7)				
1×10	391(−8)	339(−8)	227(−8)					
2								
3								
5								
7								

can be avoided, if data from more than one observation well are available, by preparing a composite data plot of s versus t/r^2. This data plot would be matched by adding the constraint that the r values for the different data curves representing each well fall on proportional β curves.

The "late" data (for large values of t) can be analyzed using equations 16, 17, and 18; these equations are forms of summaries 1, $W(u)$, and 4, $L(u, v)$. However, for cases 1 and 3, the late data fall on the flat part of the $L(u,v)$ curves and a time-drawdown plot match would be indeterminate. Thus, only a distance-drawdown match could be used. Drawdown predictions, however, could be made using the $L(u, v)$ curves.

Assumption 5, that no drawdown occurs in the source beds, has been examined by Neuman and Witherspoon (1969a, p. 810, 811) for the situation in which two aquifers are separated by a less permeable bed. This is equivalent to case 3 with $K''=0$ and $S''=0$. They concluded that (1) $H(u,\beta)$, in the asymptotic solution for early times, would not be affected appreciably because the properties of the source bed have a negligible effect on the solution for $Tt/r^2S \leq 1.6\beta^2/(r/B)^4$, which is equivalent to $t \leq S'b'/10K'$, where $B=\sqrt{Tb'/K'}$; and (2) if $T_s > 100T$, where T_s represents the transmissivity of the source bed, it is probably justified to neglect drawdown in the unpumped aquifer.

Table 5.2 is a listing of a FORTRAN program for computing values of $H(u,\beta)$ for $u \geq 10^{-60}$ using a procedure devised and programed by S. S. Papadopulos. Input data for this program consists of three cards. The first card contains the beginning value of $1/u$, coded in columns 1–10, in format E10.5, and the ending (largest) value of $1/u$, coded in columns 11–20, in format E10.5. The next two cards contain 12 values of β, coded in columns 1–10, 11–20, . . . , and 71–80 on the first card and columns 1–10, 11–20, . . . , 31–40 on the second card, all in format E10.5. The function is evaluated as follows (S. S. Papadopulos, written commun., 1975):

$$H(u,\beta) = \int_u^\infty (e^{-y}/y)\ \mathrm{erfc}\ (\beta\sqrt{u}/\sqrt{y(y-u)})\ dy$$

$$= \int_u^\infty f\, dy\ ,$$

where f represents the integrand. For $\beta=0$, $H(u,\beta)=W(u)$, where $W(u)$ is the well function of Theis. Because $\mathrm{erfc}(x) \leq 1$ for $x \geq 0$, it follows that $H(u,\beta) \leq W(u)$, and for $u>10$, $W(u) \approx 0$ and therefore for $u>10$, $H(u,\beta) \approx 0$. The tables of $H(u,\beta)$ indicate that $H(u,\beta) \approx 0$ for $\beta > 1$ and $\beta^2 u > 300$. For an arbitrarily small value of u, the integral can be considered as the sum of three integrals

$$\int_u^\infty f\, dy = \int_u^{u_1} f\, dy + \int_{u_1}^{u_2} f\, dy + \int_{u_2}^\infty f\, dy\ ,$$

where $u_2 = (u/2)(1 + \sqrt{1+10^{20}\beta^2/u})$,

and $u_1 = (u/2)(1 + \sqrt{1+0.025\ \beta^2/u})$.

The significance of u_2 and u_1 is that

$$\mathrm{erfc}\ (\beta\sqrt{u}/\sqrt{y(y-u)}) \approx 1 \text{ for } u>u_2$$

and

$$\mathrm{erfc}\ (\beta\sqrt{u}/\sqrt{y(y-u)}) \approx 0 \text{ for } u<u_1.$$

Therefore,

$$\int_u^{u_1} f\, dy \approx 0\ ,$$

and

$$\int_{u_2}^\infty f\, dy \approx W(u_2),$$

where $W(u_2)$ is the well function of Theis. The function can be evaluated as

$$H(u,\beta) \approx W(u) \text{ for } u > u_2$$

$$H(u,\beta) \approx \int_u^{u_2} f\, dy + W(u_2) \text{ for } u_1 < u < u_2$$

and $$H(u,\beta) \approx \int_{u_1}^{u_2} f\, dy + W(u_2) \text{ for } u < u_1\ .$$

If $u_2 > 10$, then

$$\int_{u_1}^{u_2} f\, dy = \int_{u_1}^{10} f\, dy,\ W(u_2) \approx 0\ .$$

An example of output from this program is shown in figure 5.3.

Solution 6: Constant discharge from a partially penetrating well in a leaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.

```
H(U,BETA)

              | BETA
        1/U   | 0.30E-01    0.10E 00    0.30E 00    0.10E 01    0.30E 01
  0.100E 02     1.6667      1.3894      0.9358      0.3970      0.0966
  0.150E 02     1.9953      1.6531      1.1203      0.5010      0.1379
  0.200E 02     2.2308      1.8401      1.2536      0.5812      0.1733
  0.300E 02     2.5626      2.1010      1.4435      0.7023      0.2320
  0.500E 02     2.9759      2.4228      1.6853      0.8677      0.3214
  0.700E 02     3.2428      2.6296      1.8457      0.9836      0.3897
  0.100E 03     3.5196      2.8443      2.0164      1.1122      0.4702
  0.150E 03     3.8256      3.0826      2.2112      1.2647      0.5717
  0.200E 03     4.0369      3.2483      2.3499      1.3767      0.6498
  0.300E 03     4.3259      3.4775      2.5459      1.5394      0.7683
  0.500E 03     4.6754      3.7598      2.7938      1.7516      0.9305
  0.700E 03     4.8969      3.9425      2.9576      1.8953      1.0447
  0.100E 04     5.1247      4.1338      3.1317      2.0507      1.1715
  0.150E 04     5.3756      4.3486      3.3301      2.2306      1.3225
  0.200E 04     5.5488      4.4996      3.4712      2.3602      1.4335
  0.300E 04     5.7871      4.7109      3.6704      2.5452      1.5951
  0.500E 04     6.0787      4.9747      3.9220      2.7819      1.8062
  0.700E 04     6.2665      5.1474      4.0880      2.9396      1.9494
  0.100E 05     6.4623      5.3297      4.2643      3.1082      2.1042
  0.150E 05     6.6816      5.5361      4.4650      3.3014      2.2837
  0.200E 05     6.8353      5.6821      4.6076      3.4394      2.4131
  0.300E 05     7.0498      5.8874      4.8087      3.6349      2.5979
  0.500E 05     7.3170      6.1454      5.0624      3.8827      2.8344
  0.700E 05     7.4915      6.3149      5.2297      4.0467      2.9921
  0.100E 06     7.6754      6.4944      5.4072      4.2212      3.1606
  0.150E 06     7.8834      6.6983      5.6090      4.4202      3.3538
  0.200E 06     8.0304      6.8427      5.7523      4.5617      3.4917
  0.300E 06     8.2369      7.0462      5.9544      4.7616      3.6872
  0.500E 06     8.4960      7.3024      6.2091      5.0141      3.9351
  0.700E 06     8.6662      7.4710      6.3770      5.1807      4.0991
  0.100E 07     8.8463      7.6497      6.5549      5.3576      4.2736
  0.150E 07     9.0507      7.8528      6.7573      5.5589      4.4726
  0.200E 07     9.1955      7.9968      6.9010      5.7018      4.6141
  0.300E 07     9.3995      8.1998      7.1034      5.9035      4.8141
  0.500E 07     9.6560      8.4554      7.3586      6.1578      5.0666
  0.700E 07     9.8249      8.6237      7.5267      6.3255      5.2332
  0.100E 08    10.0038      8.8022      7.7049      6.5033      5.4101
  0.150E 08    10.2070      9.0050      7.9075      6.7055      5.6114
  0.200E 08    10.3512      9.1489      8.0512      6.8490      5.7544
  0.300E 08    10.5543      9.3517      8.2539      7.0513      5.9561
  0.500E 08    10.8101      9.6072      8.5092      7.3063      6.2104
  0.700E 08    10.9785      9.7754      8.6773      7.4744      6.3781
  0.100E 09    11.1570      9.9538      8.8556      7.6525      6.5559
  0.150E 09    11.3599     10.1566      9.0583      7.8550      6.7581
  0.200E 09    11.5039     10.3004      9.2021      7.9988      6.9016
  0.300E 09    11.7067     10.5032      9.4048      8.2014      7.1040
  0.500E 09    11.9622     10.7586      9.6602      8.4566      7.3590
  0.700E 09    12.1305     10.9269      9.8284      8.6248      7.5270
  0.100E 10    12.3089     11.1052     10.0067      8.8031      7.7052
```

FIGURE 5.3.—Example of output from program for computing drawdown due to constant discharge from a well in a leaky aquifer with storage of water in the confining beds.

4. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
5. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
6. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
7. Flow is vertical in the confining bed.
8. The leakage from the confining bed is assumed to be generated within the aquifer so that in the aquifer no vertical flow results from leakage alone.

Differential equation:

$$\partial^2 s/\partial r^2 + 1/r\ \partial s/\partial r + a^2 \partial^2 s/\partial z^2 - sK'/Tb' = S/T\ \partial s/\partial t$$

$$a^2 = K_z/K_r$$

This is the differential equation describing nonsteady radial and vertical flow in a homogeneous aquifer with radial-vertical anisotropy and leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,z,0)=0,\ r \geqslant 0,\ 0 \leqslant z \leqslant b \qquad (1)$$

$$s(\infty,z,t)=0,\ 0 \leqslant z \leqslant b,\ t \geqslant 0 \qquad (2)$$

$$\partial s(r,0,t)/\partial z = 0,\ r \geqslant 0,\ t \geqslant 0 \qquad (3)$$

$$\partial s(r,b,t)/\partial z = 0,\ r \geqslant 0,\ t \geqslant 0 \qquad (4)$$

$$\lim_{r \to 0} r \frac{\partial s}{\partial r} = \begin{cases} 0, & \text{for } 0 < z < d \\ -Q/(2\pi K_r(l-d)), & \text{for } d < z < l \\ 0, & \text{for } l < z < b \end{cases} \qquad (5)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen and not by leakage. Equation 5 states that near the pumping well the discharge is distributed uniformly over the well screen and that no radial flow occurs above and below the screen.

Solution:

I. For the drawdown in a piezometer, a solution by Hantush (1964a, p. 350) is given by

$$s = Q/4\pi T\{W(u,\beta) + f(u,ar/b,\beta,d/b,l/b,z/b)\},$$

where

$$W(u,\beta) = \int_u^\infty \frac{e^{-y-\frac{\beta^2}{4y^2}}}{y}\,dy$$

$$u = \frac{r^2 S}{4Tt}$$

$$\beta = \sqrt{\frac{r^2 K'}{Tb'}}$$

$$a = \sqrt{K_z/K_r}$$

$$f(u,ar/b,\beta,d/b,l/b,z/b) = 2b/\pi(l-d) \sum_{n=1}^{\infty} l/n(\sin n\pi l/b - \sin n\pi d/b) \cdot \cos(n\pi z/b) W\left(u, \sqrt{\beta^2 + (n\pi ar/b)^2}\right).$$

II. For the drawdown in an observation well

$$s = Q/4\pi T\{W(u,\beta) + \bar{f}(u,ar/b,\beta,d/b,l/b,d'/b,l'/b)\},$$

where

$$\bar{f}(u,ar/b,\beta,d/b,l/b,d'/b,l'/b) = 2b^2/\pi^2(l-d)(l'-d') \cdot \sum_{n=1}^{\infty} 1/n^2(\sin n\pi l/b - \sin n\pi d/b) \cdot (\sin n\pi l'/b - \sin n\pi d'/b) W(u, \sqrt{\beta^2 + (n\pi ar/b)^2})$$

Comments:

The geometry is shown in figure 6.1. The differential equation and boundary conditions are based on the assumption that vertical flow in the aquifer is caused by partial penetration of the pumping well and not by leakage. Hantush (1967, p. 587) concluded that this assumption is correct if $b\sqrt{K'/Tb'} < 0.1$. The solutions are based on a uniform distribution of flow over the screen of the pumped well. Depending on friction losses within the well, a more realistic assumption might be constant drawdown over

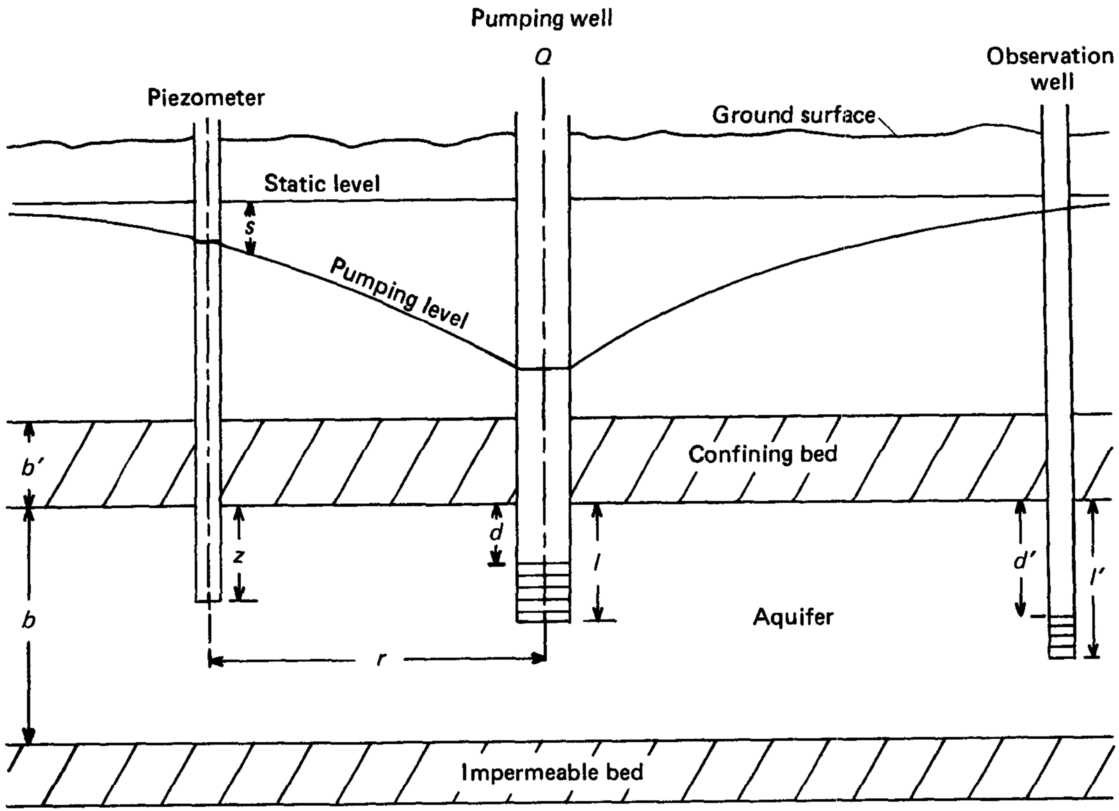

FIGURE 6.1.—Cross section through a discharging well that is screened in part of a leaky aquifer.

the screen of the pumped well; this assumption would imply nonuniform distribution of flow. Hantush (1964a, p. 351) postulates that the actual drawdown at the face of the pumping well will have a value between these two extremes. The solutions should be applied with caution at locations very near the pumped well. The effects of partial penetration are insignificant for $r>1.5\ b/a$ (Hantush, 1964a, p. 350), and the solution is the same for the solution 4.

Because of the large number of variables involved, presentation of a complete set of type curves is impractical. An example, consisting of curves for selected values of the parameters, is shown in figure 6.2 on plate 1. This figure is based on function values generated by a FORTRAN program.

The computer program formulated to compute drawdowns due to pumping a partially penetrating well in a leaky aquifer is listed in table 6.1. Input data to this program consists of cards coded in specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: aquifer thickness (b), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of pumping well screen (l), coded in format F5.1 in columns 6–10; depth, below top of aquifer, to top of pumping well screen (d), coded in format F5.1 in columns 11–15; number of observation wells and piezometers, coded in format I5 in columns 16–20; smallest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 21–30; largest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 31–40. The next two cards contain 12 values of r/B, all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Computation will terminate with the first zero (or blank) value coded. Next is a series of cards, one card per observation well or piezometer, containing: radial distance from the pumped well multiplied by the square root of the ratio of vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of observation well screen (code blank for piezometer), coded in format F5.1, in columns 6–10; depth, below top of aquifer, to top of observation well screen (total depth for a piezometer), coded in format F5.1, in columns 11–15. Output from this program is a table of function values. An example of the output is shown in figure 6.3.

Because most aquifers are anisotropic in the $r-z$ plane, it is generally impractical to use this solution to analyze for the parameters. However, it can be used to predict drawdown if the parameters are determined independently.

```
W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,Z/B), Z/B= 0.50, SORT(KZ/KR)*R/B= 0.10, L/B= 0.70, D/B= 0.30

         | R/BR
     1/U |  0.10E-05  0.10E-04  0.10E-03  0.10E-02  0.10E-01  0.10E 00  0.10E 01  0.10E 02
0.100E 01     0.5478    0.5478    0.5478    0.5478    0.5478    0.5468    0.4631    0.0001
0.150E 01     0.9901    0.9901    0.9901    0.9901    0.9900    0.9878    0.7872    0.0001
0.200E 01     1.3804    1.3804    1.3804    1.3804    1.3803    1.3764    1.0398    0.0001
0.300E 01     2.0043    2.0043    2.0043    2.0043    2.0042    1.9964    1.3767    0.0001
0.500E 01     2.8381    2.8381    2.8381    2.8381    2.8379    2.8221    1.6931    0.0001
0.700E 01     3.3737    3.3737    3.3737    3.3737    3.3735    3.3499    1.8158    0.0001
0.100E 02     3.9049    3.9049    3.9049    3.9049    3.9046    3.8700    1.8826    0.0001
0.150E 02     4.4488    4.4488    4.4488    4.4488    4.4483    4.3975    1.9094    0.0001
0.200E 02     4.7951    4.7951    4.7951    4.7951    4.7944    4.7291    1.9143    0.0001
0.300E 02     5.2379    5.2379    5.2379    5.2379    5.2369    5.1455    1.9155    0.0001
0.500E 02     5.7539    5.7539    5.7539    5.7539    5.7525    5.6135    1.9155    0.0001
0.700E 02     6.0864    6.0864    6.0864    6.0864    6.0844    5.9001    1.9155    0.0001
0.100E 03     6.4390    6.4390    6.4390    6.4389    6.4363    6.1859    1.9155    0.0001
0.150E 03     6.8411    6.8411    6.8411    6.8411    6.8372    6.4816    1.9155    0.0001
0.200E 03     7.1271    7.1271    7.1271    7.1271    7.1220    6.6669    1.9155    0.0001
0.300E 03     7.5309    7.5309    7.5309    7.5309    7.5233    6.8854    1.9155    0.0001
0.500E 03     8.0404    8.0404    8.0404    8.0403    8.0278    7.0788    1.9155    0.0001
0.700E 03     8.3763    8.3763    8.3763    8.3762    8.3588    7.1556    1.9155    0.0001
0.100E 04     8.7326    8.7326    8.7326    8.7323    8.7076    7.2002    1.9155    0.0001
0.150E 04     9.1377    9.1377    9.1377    9.1373    9.1005    7.2199    1.9155    0.0001
0.200E 04     9.4252    9.4252    9.4252    9.4247    9.3758    7.2239    1.9155    0.0001
0.300E 04     9.8305    9.8305    9.8305    9.8298    9.7568    7.2250    1.9155    0.0001
0.500E 04    10.3412   10.3412   10.3412   10.3400   10.2199    7.2251    1.9155    0.0001
0.700E 04    10.6776   10.6776   10.6776   10.6759   10.5099    7.2251    1.9155    0.0001
0.100E 05    11.0343   11.0343   11.0343   11.0318   10.7990    7.2251    1.9155    0.0001
W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,L'/B,D'/B), L'/B= 0.51, D'/B= 0.49, SQRT(KZ/KR)*R/B= 0.10,
  L/B= 0.70, D/H= 0.30
         | R/BR
     1/U |  0.10E-05  0.10E-04  0.10E-03  0.10E-02  0.10E-01  0.10E 00  0.10E 01  0.10E 02
0.100E 01     0.5477    0.5477    0.5477    0.5477    0.5477    0.5468    0.4631    0.0001
0.150E 01     0.9899    0.9899    0.9899    0.9899    0.9899    0.9876    0.7871    0.0001
0.200E 01     1.3801    1.3801    1.3801    1.3801    1.3801    1.3761    1.0396    0.0001
0.300E 01     2.0038    2.0038    2.0038    2.0038    2.0037    1.9959    1.3764    0.0001
0.500E 01     2.8372    2.8372    2.8372    2.8372    2.8371    2.8213    1.6927    0.0001
0.700E 01     3.3727    3.3727    3.3727    3.3727    3.3725    3.3488    1.8153    0.0001
0.100E 02     3.9037    3.9037    3.9037    3.9037    3.9034    3.8688    1.8821    0.0001
0.150E 02     4.4475    4.4475    4.4475    4.4475    4.4470    4.3962    1.9089    0.0001
0.200E 02     4.7937    4.7937    4.7937    4.7937    4.7930    4.7277    1.9138    0.0001
0.300E 02     5.2365    5.2365    5.2365    5.2365    5.2356    5.1441    1.9150    0.0001
0.500E 02     5.7525    5.7525    5.7525    5.7525    5.7511    5.6122    1.9150    0.0001
0.700E 02     6.0850    6.0850    6.0850    6.0849    6.0830    5.8987    1.9150    0.0001
0.100E 03     6.4376    6.4376    6.4376    6.4375    6.4349    6.1845    1.9150    0.0001
0.150E 03     6.8397    6.8397    6.8397    6.8397    6.8358    6.4802    1.9150    0.0001
0.200E 03     7.1257    7.1257    7.1257    7.1257    7.1206    6.6655    1.9150    0.0001
0.300E 03     7.5295    7.5295    7.5295    7.5295    7.5219    6.8840    1.9150    0.0001
0.500E 03     8.0390    8.0390    8.0390    8.0389    8.0264    7.0775    1.9150    0.0001
0.700E 03     8.3749    8.3749    8.3749    8.3748    8.3574    7.1542    1.9150    0.0001
0.100E 04     8.7312    8.7312    8.7312    8.7309    8.7062    7.1988    1.9150    0.0001
0.150E 04     9.1363    9.1363    9.1363    9.1359    9.0991    7.2185    1.9150    0.0001
0.200E 04     9.4238    9.4238    9.4238    9.4233    9.3743    7.2225    1.9150    0.0001
0.300E 04     9.8291    9.8291    9.8291    9.8284    9.7554    7.2236    1.9150    0.0001
0.500E 04    10.3398   10.3398   10.3398   10.3386   10.2185    7.2237    1.9150    0.0001
0.700E 04    10.6762   10.6762   10.6762   10.6745   10.5085    7.2237    1.9150    0.0001
0.100E 05    11.0329   11.0329   11.0328   11.0304   10.7976    7.2237    1.9150    0.0001
```

FIGURE 6.3.—Example of output from program for partial penetration in a leaky artesian aquifer.

Solution 7: Constant drawdown in a well in a leaky aquifer

Assumptions:

1. Water level in well is changed instantaneously by s_w at $t=0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\partial^2 s/\partial r^2 + (1/r)\partial s/\partial r - sK'/Tb' = (S/T)\partial s/\partial t$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic confined aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,0)=0,\ r\geqslant 0 \qquad (1)$$

$$s(r_w,t)=s_w,\ t\geqslant 0 \qquad (2)$$

$$s(\infty,t)=0,\ t\geqslant 0 \qquad (3)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that at the wall or screen of the discharging well, drawdown in the aquifer is equal to the constant drawdown in the well, which assumes that there is no entrance loss to the discharging well. Equation 3 states that the drawdown approaches zero as distance from the discharging well approaches infinity.

Solutions (Hantush, 1959):

I. For the discharge rate of the well,

$$Q = 2\pi T s_w G(\alpha, r_w/B),$$

where

$$G(\alpha,r_w/B) = (r_w/B)K_1(r_w/B)/K_0(r_w/B) + (4/\pi^2)\exp\left[-\alpha(r_w/B)^2\right] \cdot \int_0^\infty \left\{u\exp(-\alpha u^2)/\left[J_0^2(u) + Y_0^2(u)\right]\right\} \cdot du/\left[u^2 + (r_w/B)^2\right],$$

and

$$\alpha = Tt/Sr_w^2,$$

$$B = \sqrt{Tb'/K'}.$$

K_0 and K_1 are zero-order and first-order, respectively, modified Bessel functions of the second kind. J_0 and Y_0 are the zero-order Bessel functions of the first and second kind, respectively.

II. For the drawdown in water level

$$s = s_w(K_0(r/B)/K_0(r_w/B) + (2/\pi)\exp(-\alpha r_w^2/B^2)\int_0^\infty \frac{\exp(-\alpha u^2)}{u^2 + (r_w/B)^2} \cdot \frac{J_0(ur/r_w)Y_0(u) - Y_0(ur/r_w)J_0(u)}{J_0^2(u) + Y_0^2(u)}\, u\, du \qquad (4)$$

with α, B, K_0, J_0, and Y_0 as defined previously.

Comments:

A cross section through the discharging well is shown in figure 7.1. The boundary conditions most commonly apply to a flowing artesian well, as is shown in this illustration.

Figure 7.2 on plate 1 is a plot of dimensionless discharge ($G(\alpha,r_w/B)$) versus dimensionless time (α) from data of Hantush (1959, table 1) and Dudley (1970, table 2). Selected values of $G(\alpha,r_w/B)$ are given in table 7.1. The corresponding data curve should be a plot of observed discharge versus time. The data curve is matched to figure 7.2 and from match points ($\alpha,G(\alpha,r_w/B)$) and (t,Q), T and S are computed from the equations

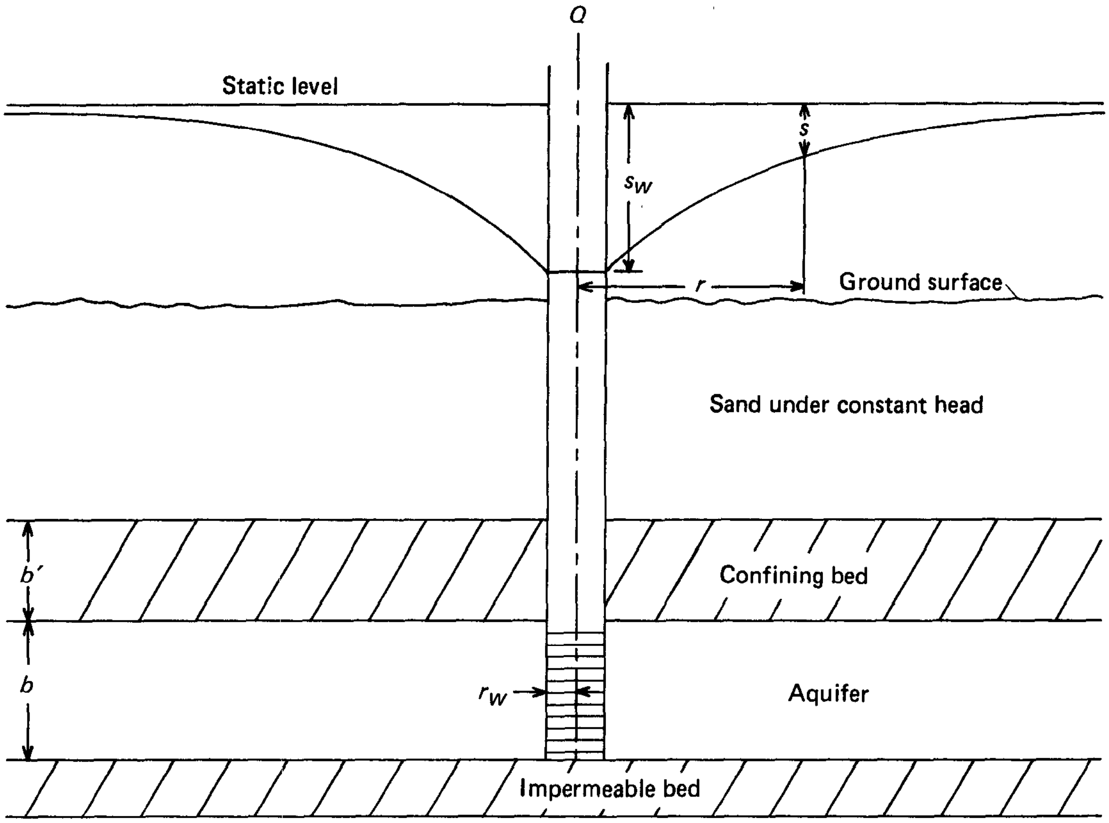

FIGURE 7.1.—Cross section through a well with constant drawdown in a leaky aquifer.

$$T = Q/(2\pi s_w G(\alpha, r_w/B))$$

and $$S = Tt/(\alpha r_w^2).$$

Figure 7.3 on plate 1 contains plots of dimensionless drawdown (s/s_w) versus dimensionless time ($\alpha r_w^2/r^2$). The corresponding data plot would be observed drawdown versus observation time. Matching the data and type curves by superposition and choosing convenient match points ($s/s_w, \alpha r_w^2/r^2$) and (s,t), the ratio of transmissivity to storage coefficient can be computed from the relation

$$T/S = (\alpha r_w^2/r^2)(r^2/t).$$

Figure 7.3 was plotted from function values generated by a FORTRAN program. This program is listed in table 7.2. The input data for this program consist of three cards coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of alpha for which computation is desired, coded in format E10.5 in columns 1–10; the largest value of alpha for which computation is desired, coded in format E10.5 in columns 11–20. The output table will include a range in alpha spanning these two values up to a limiting range of nine log cycles. The second card contains 13 values of r_w/B. These coded values are the significant figures only and should be greater or equal to 1 and less than 10. The power of 10 by which each of these coded values is multiplied is calculated by the program. Zero (or blank) coding is permissible, but the first zero (or blank) value will terminate the list. The 13 values, all coded in format F5.0, are coded in columns 1–5, 6–10, 11–15, 16–20, 21–25, 26–30, 31–35, 36–40, 41–45, 46–50, 51–55, 56–60, and 61–65. The third card contains the radius of the control well and distances to the observation wells.

TABLE 7.1.—*Values of $G(\alpha, r_w/B)$*

[Values for $r_w/B \leqslant 1\times10^{-2}$ and $\alpha \geqslant 1\times10^{2}$ are from Hantush (1959, table 1), others are from Dudley (1970, table 2)]

α	r_w/B								
	0	6×10^{-3}	1×10^{-2}	2×10^{-2}	6×10^{-2}	1×10^{-1}	2×10^{-1}	6×10^{-1}	1×10^{0}
1 $\times 10^{-1}$	2.24	2.24	2.24	2.25	2.25	2.25	2.26	2.31	2.43
2	1.71	1.71	1.71	1.71	1.72	1.72	1.73	1.81	1.96
5	1.23	1.23	1.23	1.23	1.23	1.24	1.25	1.38	1.61
1 $\times 10^{0}$	.983	.983	.983	.984	.986	.990	1.01	1.18	1.49
2	.800	.800	.800	.801	.804	.809	.834	1.07	1.44
5	.628	.628	.628	.629	.633	.642	.682	1.01	1.43
1 $\times 10^{1}$	.534	.534	.534	.535	.541	.554	.611		
2	.461	.461	.461	.462	.472	.491	.569		
5	.389	.389	.389	.390	.407	.438	.548		
1 $\times 10^{2}$	.346	.346	.346	.349	.374	.417	.545		
2	.311	.311	.312	.316	.353	.408			
5	.274	.275	.276	.284	.341	.406			
1 $\times 10^{3}$	.251	.252	.255	.266	.339				
2	.232	.234	.239	.255					
5	.210	.215	.222	.249					
1 $\times 10^{4}$	.196	.204	.216	.248					
2	.185	.197	.213						
5	.170	.192	.212						
1 $\times 10^{5}$	.161	.191							
2	.152								
5	.143								
1 $\times 10^{6}$	.136								
2	.130								
5	.123	.191	.212	.248	.339	.406	.545	1.01	1.43

α	r_w/B								
	0	1×10^{-5}	2×10^{-5}	6×10^{-5}	1×10^{-4}	2×10^{-4}	6×10^{-4}	1×10^{-3}	2×10^{-3}
1 $\times 10^{4}$	0.196	0.196	0.196	0.196	0.196	0.196	0.196	0.196	0.197
2	.185	.185	.185	.185	.185	.185	.185	.185	.185
5	.170	.170	.170	.170	.170	.170	.170	.170	.173
1 $\times 10^{5}$	.161	.161	.161	.161	.161	.161	.162	.162	.167
2	.152	.152	.152	.152	.152	.152	.153	.155	.163
5	.143	.143	.143	.143	.143	.143	.144	.148	.161
1 $\times 10^{6}$	.136	.136	.136	.136	.136	.137	.139	.144	.159
2	.130	.130	.130	.130	.130	.131	.135	.143	.159
5	.123	.123	.123	.123	.123	.124	.133	.142	.158
1 $\times 10^{7}$	.118	.118	.118	.118	.118	.120			
2	.114	.114	.114	.114	.114	.116			
5	.108	.108	.108	.108	.110				
1 $\times 10^{8}$	.104	.104	.104	.105	.108				
2	.100	.100	.101	.103	.107				
5	.0958	.0958	.0966	.102					
1 $\times 10^{9}$	.0927	.0930	.0943						
2	.0899	.0906	.0927						
5	.0864	.0880	.0916						
1 $\times 10^{10}$	.0838	.0867	.0914						
2	.0814	.0862							
5	.0785	.0860							
1 $\times 10^{11}$	.0764	.0860	.0914	.102	.107	.116	.133	.142	.158
2									
5									

The control well radius (r_w) is coded first, in columns 1–8 in format F8.2. The distances (r) to the observation wells (maximum of nine) are coded next, in monotonic increasing order (smallest r first, largest r last), in columns 9–16, 17–24, 25–32, 33–40, 41–48, 49–56, 57–64, 65–72, and 73–80, all in format F8.2. If two or more observation wells have the same distance, this common distance should be coded only once, the function values will apply to all wells at the same distance from the control well. If the number of observation wells is less than nine, the remaining columns on the card should be left blank.

The integral in equation 4 is approximated by

$$\int_0^{\infty} f(u,\alpha,r_w/B)\,du \doteq \sum_{i=1}^{8000} f(-\Delta u/2 + i\Delta u,\alpha,r_w/B)\,\Delta u\,.$$

This expression is a composite quadrature with equally spaced abscissas. The abscissas are chosen at the midpoints of the intervals instead of the ends because the integrand is singular at $u=0$. The value of Δu used is related to α and is $\Delta u \leqslant 10^{-3}/\sqrt{\alpha}$. The r_w/B values then selected by the program satisfy $r_w/B \geqslant 10\ \Delta u$. These two constraints, though empirical, are related to the behavior of the integrand; the first constraint is related to the term $e^{-\alpha u^2}$ as u becomes large, and the second to $u/(u^2+(r_w/B)^2)$ as u becomes small.

The Bessel functions $K_0(r/B)$, $K_0(r_w/B)$ are evaluated by the IBM subroutine BESK. A description of this subroutine may be found in the IBM Scientific Subroutine Package.

The Bessel functions of the second kind in the integrand, $Y_0(u)$ and $Y_0(ur/r_w)$, are evaluated using IBM subroutine BESY, which is discussed in IBM SSP manual. The Bessel functions $J_0(u)$ and $J_0(ur/r_w)$ are evaluated for arguments less than four by a polynomial approximation consisting of the first 10 terms of the series expansion

$$J_0(x) = \sum_{u=0}^{\infty} (-1)^u (x^2/2)^u/(n!)^2.$$

For arguments greater than or equal to four, the asymptotic expansion is used

$$J_0(x) = P \cos (x - \pi/4) + Q \sin (x - \pi/4).$$

P and Q are calculated by the algorithm used in IBM subroutine BESY.

The output from this program consists of tables of function values, an example of which is shown in figure 7.4.

Solution 8: Constant discharge from a fully penetrating well of finite diameter in a nonleaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived from a depletion of storage in the aquifer and inside the well bore.

Differential equation:

$$\partial^2 s/\partial r^2 + (1/r)\partial s/\partial r = (S/T)\partial s/\partial t,\ r \geqslant r_w$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer in the region outside the pumped well.

Boundary and initial conditions:

$$s(r_w, t) = s_w(t),\ t>0 \qquad (1)$$

$$s(\infty, t) = 0,\ t>0 \qquad (2)$$

$$s(r, 0) = 0,\ r \geqslant r_w \qquad (3)$$

$$s_w(0) = 0 \qquad (4)$$

$$(2\pi r_w T)\partial s(r_w, t)/\partial r - (\pi r_c^2)\partial s_w(t)/\partial t = -Q,\ t>0 \qquad (5)$$

Equation 1 states that the drawdown at the well bore is equal to the drawdown inside the well, assuming that there is no entrance loss at the well face. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that, initially, drawdown in the aquifer and inside the well is zero. Equation 5 states that the discharge of the well is equal to the sum of the flow into the well and the rate of decrease in storage inside the well.

Solution (Papadopulos and Cooper, 1967; Papadopulos, 1967):

$$s = (Q/4\pi T)\, F(u,\alpha,\rho),$$

where

$$F(u,\alpha,\rho) = (8\alpha/\pi) \int_0^\infty \frac{[(1-\exp(-\beta^2\rho^2/4u)]\ [J_0(\beta\rho)A(\beta) - Y_0(\beta\rho)B(\beta)]}{\{[A(\beta)]^2 + [B(\beta)]^2\}\beta^2,}\, d\beta$$

and

$$B(\beta) = \beta J_0(\beta) - 2\alpha J_1(\beta),$$

$$A(\beta) = \beta Y_0(\beta) - 2\alpha Y_1(\beta),$$

$$u = r^2 S/4Tt,$$

$$\alpha = r_w^2 S/r_c^2,$$

and

$$\rho = r/r_w.$$

J_0 and Y_0, J_1 and Y_1, are zero-order and first-order Bessel functions of the first and second kind, respectively.

```
Z(ALPHA,R/RW,RW/B), R/RW=  100.

          I RW/B
  ALPHA   I 0.10E-03 0.15E-03 0.20E-03 0.30E-03 0.50E-03 0.70E-03 0.10E-02 0.15E-02 0.20E-02 0.30E-02 0.50E-02 0.70E-02 0.10E-01
 0.100E 05     0.114    0.114    0.114    0.114    0.113    0.113    0.113    0.112    0.112    0.109    0.102    0.091    0.074
 0.150E 05     0.142    0.142    0.142    0.141    0.141    0.141    0.141    0.140    0.138    0.134    0.122    0.107    0.082
 0.200E 05     0.161    0.161    0.161    0.161    0.161    0.161    0.160    0.159    0.157    0.151    0.135    0.115    0.086
 0.300E 05     0.188    0.188    0.188    0.188    0.188    0.188    0.187    0.184    0.181    0.173    0.150    0.123    0.088
 0.500E 05     0.221    0.221    0.221    0.221    0.220    0.220    0.218    0.214    0.209    0.196    0.162    0.128    0.089
 0.700E 05     0.242    0.242    0.242    0.241    0.241    0.240    0.237    0.232    0.225    0.208    0.167    0.130    0.089
 0.100E 06     0.263    0.262    0.262    0.262    0.261    0.260    0.257    0.250    0.240    0.218    0.169    0.130    0.089
 0.150E 06     0.285    0.285    0.285    0.284    0.283    0.281    0.277    0.267    0.254    0.225    0.170    0.130    0.089
 0.200E 06     0.300    0.300    0.300    0.299    0.298    0.295    0.289    0.277    0.262    0.228    0.171    0.130    0.089
 0.300E 06     0.321    0.321    0.320    0.319    0.317    0.313    0.305    0.289    0.269    0.231    0.171    0.130    0.089
 0.500E 06     0.345    0.345    0.344    0.343    0.339    0.333    0.322    0.299    0.275    0.232    0.171    0.130    0.089
 0.700E 06     0.360    0.360    0.359    0.357    0.352    0.344    0.330    0.303    0.276    0.232    0.171    0.130    0.089
 0.100E 07     0.375    0.375    0.374    0.371    0.364    0.355    0.337    0.305    0.277    0.232    0.171    0.130    0.089
 0.150E 07     0.391    0.391    0.389    0.386    0.376    0.364    0.342    0.306    0.277    0.232    0.171    0.130    0.089
 0.200E 07     0.402    0.401    0.400    0.396    0.384    0.368    0.344    0.307    0.277    0.232    0.171    0.130    0.089
 0.300E 07     0.417    0.416    0.414    0.408    0.392    0.373    0.345    0.307    0.277    0.232    0.171    0.130    0.089
 0.500E 07     0.435    0.432    0.429    0.421    0.399    0.376    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.700E 07     0.445    0.442    0.438    0.427    0.401    0.376    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.100E 08     0.456    0.452    0.446    0.433    0.403    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.150E 08     0.467    0.461    0.454    0.437    0.403    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.200E 08     0.474    0.467    0.458    0.439    0.404    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.300E 08     0.483    0.473    0.462    0.440    0.404    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.500E 08     0.492    0.479    0.465    0.440    0.404    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.700E 08     0.497    0.482    0.466    0.440    0.404    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
 0.100E 09     0.501    0.483    0.467    0.440    0.404    0.377    0.346    0.307    0.277    0.232    0.171    0.130    0.089
```

FIGURE 7.4.—Example of output from program for constant drawdown in a well in a leaky artesian aquifer.

The drawdown inside the pumped well is obtained at $r = r_w$ and can be expressed as (Papadopulos and Cooper, 1967, p. 242):

$$s_w = (Q/4\pi T)\,F(u_w,\alpha),$$

where $$F(u_w,\alpha) = F(u,\alpha,1),$$

and $$u_w = r_w^2 S/4tT.$$

Comments: A cross section through the discharging well is shown in figure 8.1. The geometry, except for the region of the well bore, is the same as for solution 1 (Theis solution). It is apparent from figure 8.2 and 8.3 (on plate 1) that $F(u,\alpha,\rho)$ approaches $W(u)$, the Theis solution, as time becomes large. Papadopulos (1967, p. 161) stated that for $t > 2.5\times10^3 r_c/T$, or $\alpha\rho^2/u > 10^4$, the function $F(u,\alpha,\rho)$ can be closely approximated by $F(u,\alpha,\rho)=W(u)$. Papadopulos and Cooper (1967, p. 242) stated that for $t > 2.5\times10^2\ r_c^2/T$, or $\alpha/u_w > 10^3$, the function $F(u_w,\alpha)$ can be closely approximated by $F(u_w,\alpha)=W(u_w)$. An examination of the type curves and function values indicates that $F(u_w,\alpha)\approx W(u_w)$ (less than 5-percent error) for $\alpha/u_w > 10^2$, and hence t should only be greater than $25\ r_c^2/T$ for drawdown in the pumped well.

Figures 8.2 and 8.3 were prepared from function values given in Papadopulos and Cooper (1967) and Papadopulos (1967), which are reproduced in table 8.1. For drawdown observations in the pumped well, the method of analysis is to plot drawdown versus time and

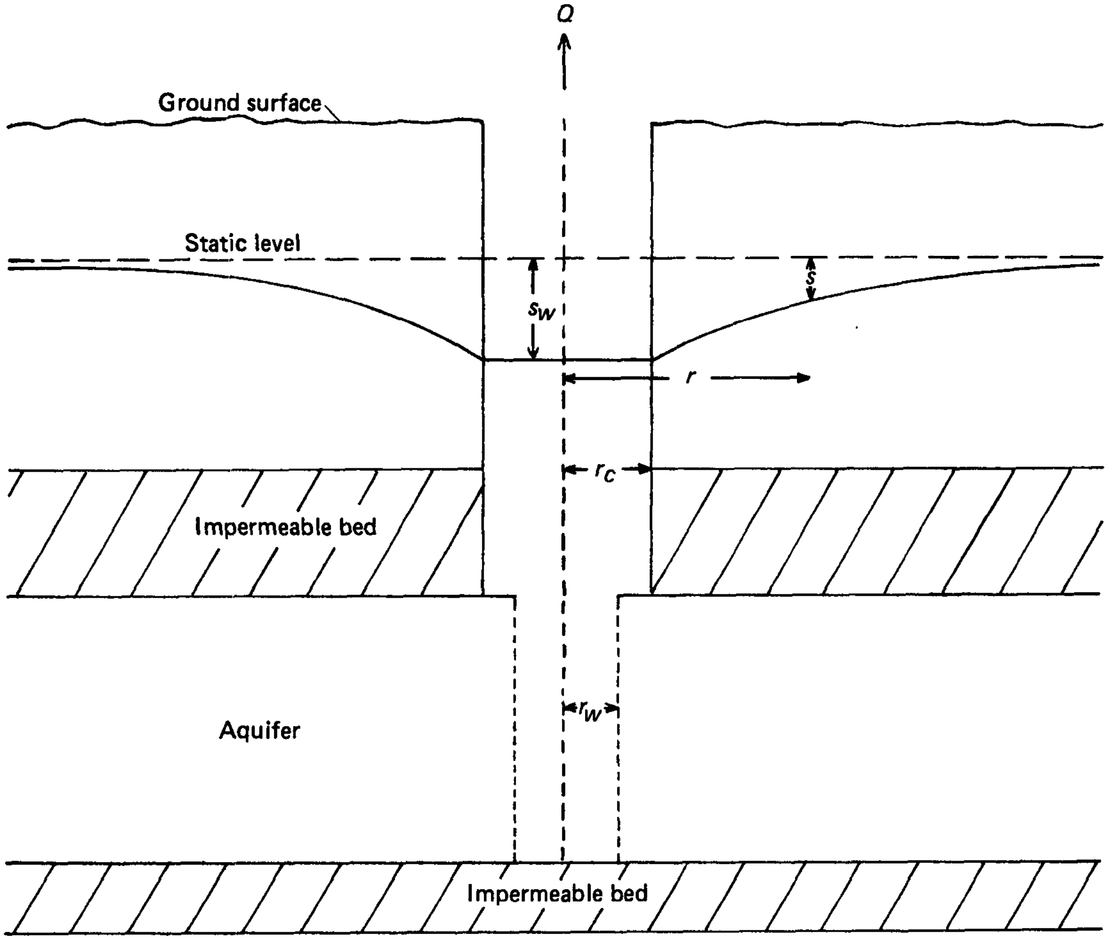

FIGURE 8.1.—Cross section through a discharging well of finite diameter.

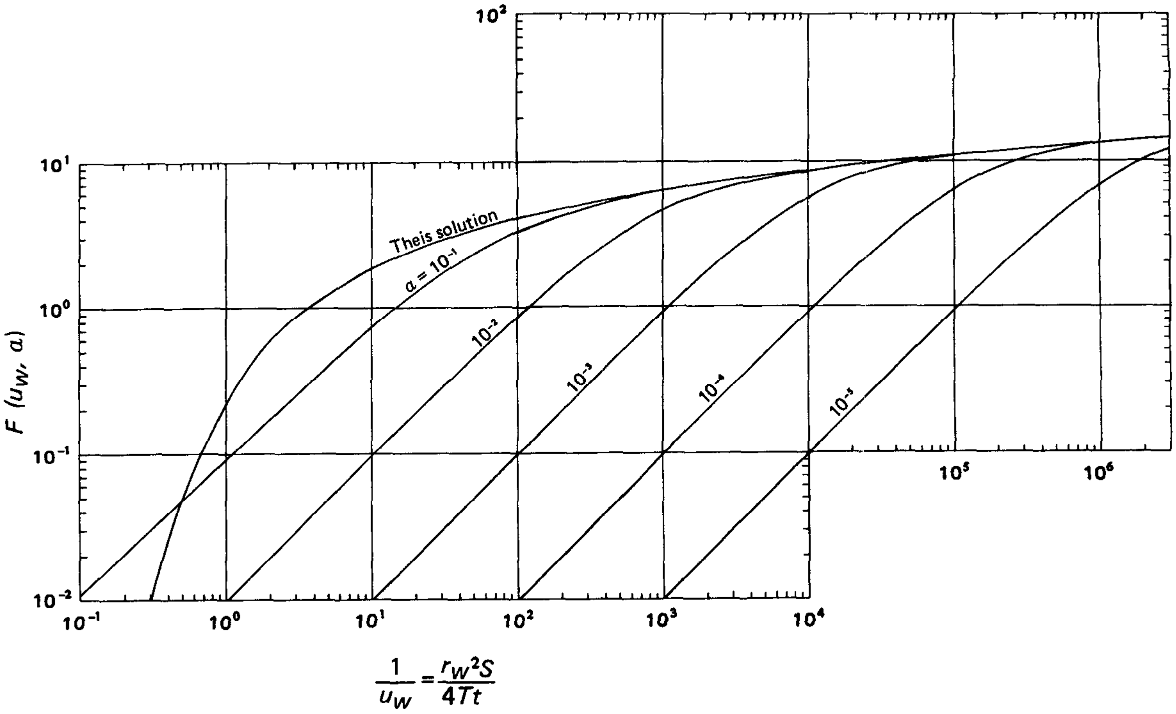

FIGURE 8.2.—Five selected type curves of $F(u_w,\alpha)$, and the Theis solution, versus $1/u_w$.

then superimpose the plot on figure 8.2. After match points of (s,t) and $(F(u_w,\alpha), 1/u_w)$ are chosen, the transmissivity can be computed from the relation $T=(Q/4\pi s)\ F(u_w,\alpha)$. Then, the storage coefficient can be determined from $S=(4Tt/r_w^2)/(1/u_w)$.

For observations not in the pumped well, two procedures are available for analyzing the data. To analyze the data from a single observation well, a family of type curves of $F(u,\alpha,\rho)$ versus $1/u$ for different values of α can be plotted for the ρ value appropriate for the observation well, using values in table 8.1. This procedure produces a family of type curves similar to that shown for $\rho=1$ in figure 8.2. If ρ for the observation well is between ρ values in table 8.1, function values can be interpolated. Using this approach, the data for the observation well are plotted as drawdown versus time and matched to the best-fitting member of the plotted type curves. Transmissivity and storage coefficient can be calculated from $T=(Q/4\pi s)$ $F(u,\alpha,\rho)$ and $S=(4Tt/r^2)/(1/u)$.

Drawdowns at more than one observation point may be combined by preparing a composite plot of the drawdowns at each observation well versus t/r^2. This composite plot would be analyzed by matching it to a family of type curves of $F(u,\alpha,\rho)$ versus $1/u$ for constant α. An example of such a type-curve family for $\alpha=10^{-4}$ is shown in figure 8.3. This method requires multiple sheets of type curves, one sheet for each value of α. When the data curves are matched to the type-curve family, care should be taken to insure that the data for each well fall on the type curve having the appropriate ρ value. This will be possible for all the data for only one value of α. Transmissivity and storage coefficient are calculated from $T=(Q/4\pi s)$ $F(u,\alpha,\rho)$ and $S=4T(t/r^2)/(1/u)$.

In both of these methods of plotting and comparing data, an alternate computation of storage coefficient is $S=r_c^2\alpha/r_w^2$. However, as pointed out by Papadopulos and Cooper (1967, p. 244), the shapes of type curves differ only slightly when α changes by an order of magnitude, therefore the determination of S is sensitive to choosing the "correct" curve. Papadopulos and Cooper (1967, p. 244) suggest that if S can be estimated within an order of magnitude, the value of α to be used for matching the data can be decided.

TABLE 8.1.—*Values of the function* $F(u,\alpha,\rho)$

[Values for ρ = 1 from Papadopulos and Cooper, 1967. Other values from Papadopulos, 1967]

u	ρ							
	1	2	5	10	20	50	100	200
For $\alpha = 10^{-1}$								
2×10^{0}	4.88×10^{-2}	1.96×10^{-2}	1.75×10^{-2}	2.41×10^{-2}	3.48×10^{-2}	4.24×10^{-2}	4.48×10^{-2}	4.50×10^{-2}
1	9.19	7.01	9.55	1.41×10^{-1}	1.85×10^{-1}	2.09×10^{-1}	2.14×10^{-1}	2.15×10^{-1}
5×10^{-1}	1.77×10^{-1}	1.95×10^{-1}	3.21×10^{-1}	4.44	5.20	5.49	5.55	5.59
2	4.06	5.78	9.42	1.13×10^{0}	1.19×10^{0}	1.22×10^{0}		
1	7.34	1.11×10^{0}	1.60×10^{0}	1.76	1.80			
5×10^{-2}	1.26×10^{0}	1.84	2.33	2.43	2.46			
2	2.30	2.97	3.28	3.34	3.35			
1	3.28	3.81	4.00	4.03				
5×10^{-3}	4.26	4.60	4.70	4.72				
2	5.42	5.58	5.63	5.64				
1	6.21	6.30	6.33					
5×10^{-4}	6.96	7.01						
2	7.87	7.93						
1	8.57	8.63						
5×10^{-5}	9.32							
2	10.24							
For $\alpha = 10^{-2}$								
2×10^{0}	4.99×10^{-3}	2.13×10^{-3}	2.11×10^{-3}	3.52×10^{-3}	7.47×10^{-3}	2.03×10^{-2}	3.44×10^{-2}	4.35×10^{-2}
1	9.91	7.99	1.32×10^{-2}	2.69×10^{-2}	6.12×10^{-2}	1.42×10^{-1}	1.91×10^{-1}	2.11×10^{-1}
5×10^{-1}	1.97×10^{-2}	2.40×10^{-2}	5.40	1.21×10^{-1}	2.63×10^{-1}	4.65	5.31	5.51
2	4.89	8.34	2.33×10^{-1}	5.12	9.15	1.16×10^{0}	1.20×10^{0}	1.22×10^{0}
1	9.67	1.93×10^{-1}	5.67	1.12×10^{0}	1.58×10^{0}	1.78	1.81	
5×10^{-2}	1.90×10^{-1}	4.16	1.18×10^{0}	1.95	2.32	2.44	2.46	
2	4.53	1.03×10^{0}	2.42	3.11	3.29	3.34	3.35	
1	8.52	1.87	3.48	3.90	4.00	4.03		
5×10^{-3}	1.54×10^{0}	3.05	4.43	4.65	4.71	4.72		
2	3.04	4.78	5.52	5.61	5.63	5.64		
1	4.55	5.90	6.27	6.31	6.33			
5×10^{-4}	6.03	6.81	6.99	7.01				
2	7.56	7.85	7.92	7.94				
1	8.44	8.59	8.63					
5×10^{-5}	9.23	9.30						
2	10.20	10.23						
1	10.87	10.93						
5×10^{-6}	11.62	11.63						
2	12.54							
1	13.24							

TABLE 8.1.—*Values of the function* $F(u,\alpha,\rho)$—Continued

u	ρ							
	1	2	5	10	20	50	100	200
				For $\alpha = 10^{-3}$				
2×10^{0}	5.00×10^{-4}	2.15×10^{-4}	2.15×10^{-4}	3.70×10^{-4}	8.35×10^{-3}	3.05×10^{-3}	8.38×10^{-3}	1.50×10^{-2}
1	9.99	8.11	1.37×10^{-3}	2.95×10^{-3}	7.58×10^{-3}	2.81×10^{-2}	7.56×10^{-2}	1.47×10^{-1}
5×10^{-1}	2.00×10^{-3}	2.45×10^{-3}	5.77	1.42×10^{-2}	3.90×10^{-2}	$1.54 \times {}^{-1}$	3.23×10^{-1}	4.78
2	4.99	8.71	2.67×10^{-2}	7.24	2.03×10^{-1}	6.59	1.02×10^{0}	1.17×10^{0}
1	9.97	2.07×10^{-2}	7.16	2.01×10^{-1}	5.41	1.38×10^{0}	1.70	1.79
5×10^{-2}	1.99×10^{-2}	4.66	1.74×10^{-1}	4.87	1.19×10^{0}	2.27	2.40	2.45
2	4.95	1.29×10^{-1}	5.05	1.31×10^{0}	2.52	3.22	3.32	3.35
1	9.83	2.70	1.04×10^{0}	2.38	3.59	3.96	4.02	
5×10^{-3}	1.95×10^{-1}	5.47	1.96	3.68	4.50	4.69	4.72	
2	4.73	1.31×10^{0}	3.81	5.23	5.55	5.63	5.64	
1	9.07	2.39	5.34	6.13	6.28	6.32		
5×10^{-4}	1.69×10^{0}	3.98	6.57	6.92	7.00	7.02		
2	3.52	6.44	7.77	7.90	7.93			
1	5.53	7.95	8.55	8.61	8.63			
5×10^{-5}	7.63	9.02	9.28	9.31				
2	9.68	10.12	10.22	10.24				
1	10.68	10.88	10.93					
5×10^{-6}	11.50	11.59	11.62					
2	12.49	12.53	12.54					
1	13.21	13.23	13.24					
5×10^{-7}	13.92	13.93						
2	14.84							
1	15.54							
				For $\alpha = 10^{-4}$				
2×10^{0}	5.00×10^{-5}	2.17×10^{-5}	2.18×10^{-5}	3.73×10^{-5}	8.46×10^{-5}	3.16×10^{-4}	9.56×10^{-4}	3.83×10^{-3}
1	1.00×10^{-4}	8.15	1.38×10^{-4}	2.98×10^{-4}	7.77×10^{-4}	3.23×10^{-3}	1.01×10^{-2}	3.42×10^{-2}
5×10^{-1}	2.00	2.47×10^{-4}	5.81	1.45×10^{-3}	4.10×10^{-3}	1.80×10^{-2}	5.62	1.75×10^{-1}
2	5.00	8.76	2.71×10^{-3}	7.54	2.27×10^{-2}	1.03×10^{-1}	3.04×10^{-1}	7.10
1	1.00×10^{-3}	2.09×10^{-3}	7.34	2.16×10^{-2}	6.69	2.97	7.92	1.43×10^{0}
5×10^{-2}	2.00	4.72	1.82×10^{-2}	5.55	1.74×10^{-1}	7.30	1.62×10^{0}	2.24
2	5.00	1.32×10^{-2}	5.56	1.74×10^{-1}	5.36	1.87×10^{-0}	2.95	3.28
1	9.98	2.81	1.23×10^{-1}	3.86	1.14×10^{0}	3.08	3.84	4.02
5×10^{-3}	1.99×10^{-2}	5.88	2.64	8.13	2.17	4.25	4.63	4.71
2	4.97	1.53×10^{-1}	6.89	1.97×10^{0}	4.14	5.47	5.60	5.63
1	9.90	3.10	1.36×10^{0}	3.44	5.61	6.24	6.31	6.33
5×10^{-4}	1.97×10^{-1}	6.18	2.53	5.26	6.71	6.98	7.01	
2	4.81	1.48×10^{0}	4.95	7.33	7.82	7.92	7.94	
1	9.34	2.72	7.03	8.37	8.57	8.62		
5×10^{-5}	1.77×10^{0}	4.65	8.65	9.20	9.29	9.32		
2	3.83	7.87	10.02	10.19	10.23	10.24		
1	6.25	9.92	10.83	10.91	10.93			
5×10^{-6}	8.99	11.23	11.57	11.62	11.63			

2	11.74	12.40	12.52	12.54				
1	12.91	13.17	13.23	13.24				
5×10^{-7}	13.78	13.90	13.93					
2	14.79	14.83						
1	15.51	15.53						
5×10^{-8}	16.22	16.23						
2	17.14							
1	17.84							

For $\alpha = 10^{-5}$

2×10^{0}	5.00×10^{-6}	2.27×10^{-6}	2.48×10^{-6}	4.19×10^{-6}	9.00×10^{-6}	3.21×10^{-5}	9.77×10^{-5}	3.15×10^{-4}
1	1.00×10^{-5}	8.36	1.44×10^{-5}	3.07×10^{-5}	7.89×10^{-5}	3.27×10^{-4}	1.04×10^{-3}	3.44×10^{-3}
5×10^{-1}	2.00	2.51×10^{-5}	5.94	1.47×10^{-4}	4.14×10^{-4}	1.84×10^{-3}	6.02	2.00×10^{-2}
2	5.00	8.87	2.74×10^{-4}	7.61	2.31×10^{-3}	1.08×10^{-2}	3.61×10^{-2}	1.19×10^{-1}
1	1.00×10^{-4}	2.11×10^{-4}	7.42	2.18×10^{-3}	6.85	3.30	1.10×10^{-1}	3.50
5×10^{-2}	2.00	4.77	1.84×10^{-3}	5.65	1.82×10^{-2}	8.90	2.92	8.57
2	5.00	1.34×10^{-3}	5.64	1.80×10^{-2}	5.92	2.89×10^{-1}	8.91	2.12×10^{0}
1	1.00×10^{-3}	2.84	1.26×10^{-2}	4.09	1.36×10^{-1}	6.49	1.80×10^{0}	3.34
5×10^{-3}	2.00	5.96	2.74	9.03	3.01	1.35×10^{0}	3.14	4.40
2	5.00	1.56×10^{-2}	7.43	2.47×10^{-1}	8.06	3.03	5.01	5.52
1	9.99	3.20	1.55×10^{-1}	5.15	1.60×10^{0}	4.75	6.06	6.27
5×10^{-4}	2.00×10^{-2}	6.54	3.20	1.04×10^{0}	2.96	6.31	6.90	6.99
2	4.98	1.66×10^{-1}	8.08	2.45	5.58	7.71	7.89	7.93
1	9.93	3.34	1.58×10^{0}	4.28	7.54	8.52	8.61	8.63
5×10^{-5}	1.98×10^{-1}	6.62	2.93	6.63	8.90	9.21	9.31	
2	4.86	1.59×10^{0}	5.86	9.36	10.10	10.22	10.24	
1	9.49	2.95	8.53	10.60	10.86	10.92		
5×10^{-6}	1.82×10^{0}	5.15	10.67	11.48	11.59	11.62		
2	4.03	9.08	12.28	12.49	12.53	12.54		
1	6.78	11.76	13.12	13.21	13.23	13.24		
5×10^{-7}	10.13	13.41	13.88	13.92	13.93			
2	13.71	14.68	14.83	14.85				
1	15.13	15.46	15.54					
5×10^{-8}	16.05	16.20						
2	17.08	17.14						
1	17.81	17.84						
5×10^{-9}	18.51							
2	19.40							
1	20.15							

The early parts (short time) of the curves in figure 8.2 are straight lines. According to Papadopulos and Cooper (1967, p. 244), these represent conditions under which all the water pumped is derived from storage within the well. The straight lines approached by the curves satisfy the equations

$$F(u_w,\alpha) = \alpha/u_w$$

and

$$s_w = Qt/\pi r_c^2 = \frac{\text{volume of water discharged}}{\text{area of well}}$$

Therefore, as pointed out by Papadopulos and Cooper (1067, p. 244), data that fall on this straight part of the type curves do not indicate information about the aquifer characteristics.

Table 8.2 is a listing of two FORTRAN programs by S. S. Papadopulos that evaluate $F(u_w,\alpha)$ and $F(u,\alpha,\rho)$. The input data to both programs consists of cards coded in specified format (readers unfamiliar with FORTRAN language format should refer to a FORTRAN language manual). Input to the programs is one or more groups of data, each group of data consisting of two cards. The first card contains one value of alpha in columns 1–10, coded in format E10.5. The program to evaluate $F(u,\alpha,\rho)$ also requires a value of rho on this card in columns 11–20. This value of rho, which must be greater than one, is also coded in format E10.5. The second card contains 16 values of u coded in columns 1–5, 6–10, . . . , 75–80 in format 16F5.0. The $F(u_w,\alpha)$ or $F(u,\alpha,\rho)$ values will be printed in the order that the u values are coded. If less than 16 values of u are desired, the remaining columns on the card may be left blank. Outputs from these two programs are shown in figures 8.4 and 8.5.

```
F(UW,ALPHA) FOR ALPHA=   1.00000E-04

      UW             INTEGRAL      INTEGRAL ERROR      F(UW,ALPHA)        X(PEAK)          Y(PEAK)

  2.00000E 00      1.54210E 03      -6.98844E-02      4.99991E-05      5.96561E-03      5.55886E 05
  1.00000E 00      3.08412E 03      -1.39817E-01      9.99956E-05      5.96561E-03      1.11177E 06
  5.00000E-01      6.16789E 03      -2.74775E-01      1.99980E-04      5.96561E-03      2.22353E 06
  2.00000E-01      1.54184E 04      -6.97533E-01      4.99907E-04      5.96561E-03      5.55875E 06
  1.00000E-01      3.08331E 04      -1.39715E 00      9.99695E-04      5.96560E-03      1.11173E 07
  5.00000E-02      6.16529E 04      -2.71364E 00      1.99896E-03      5.96559E-03      2.22335E 07
  2.00000E-02      1.54061E 05      -6.97112E 00      4.99507E-03      5.96559E-03      5.55764E 07
  1.00000E-02      3.07919E 05      -1.39383E 01      9.98359E-03      5.96554E-03      1.11128E 08
  5.00000E-03      6.15138E 05      -2.78767E 01      1.99445E-02      5.96549E-03      2.22157E 08
  2.00000E-03      1.53334E 06      -6.82757E 01      4.97152E-02      5.96527E-03      5.54652E 08
  1.00000E-03      3.05367E 06      -1.38658E 02      9.90083E-02      5.96493E-03      1.10684E 09
  5.00000E-04      6.06085E 06      -2.76458E 02      1.96509E-01      5.96425E-03      2.20389E 09
  2.00000E-04      1.48475E 07      -6.79220E 02      4.81397E-01      5.96223E-03      5.43712E 09
  1.00000E-04      2.88072E 07      -1.30780E 03      9.34008E-01      5.95886E-03      1.06380E 10
  5.00000E-05      5.45352E 07      -2.50960E 03      1.76818E 00      5.95237E-03      2.03734E 10
  2.00000E-05      1.18065E 08      -5.40026E 03      3.82800E 00      5.93415E-03      4.49196E 10
```

FIGURE 8.4.—Example of output from program for drawdown inside a well of finite diameter due to constant discharge.

```
F(U,ALPHA,RHO) FOR ALPHA=  1.00000E-05, RHO=  2.00000E 00

         U               INTEGRAL           INTEGRAL ERROR        F(U,ALPHA,RHO)

    9.99999900E-04     6.29273600E 02     5.45096700E-01     3.20486300E-02
    5.00000000E-04     1.28359500E 03     1.11649700E 00     6.53728800E-02
    1.99999900E-04     3.26376700E 03     2.47402200E 00     1.66222200E-01
    1.00000000E-04     6.55423000E 03     3.31468400E 00     3.33803700E-01
    5.00000000E-05     1.30015800E 04     3.53750700E 00     6.62164900E-01
    2.00000000E-05     3.11692500E 04     3.54940500E 00     1.58743500E 00
    9.99999900E-06     5.79505700E 04     3.54602200E 00     2.95139600E 00
    4.99999900E-06     1.01023500E 05     3.53222000E 00     5.14508300E 00
    1.99999900E-06     1.78237100E 05     3.62180400E 00     9.07753300E 00
    1.00000000E-06     2.30897600E 05     3.66347000E 00     1.17595100E 01
    4.99999900E-07     2.63222100E 05     3.68847000E 00     1.34057800E 01
    1.99999900E-07     2.88201800E 05     3.52180300E 00     1.46779900E 01
```

FIGURE 8.5.—Example of output from program for drawdown outside a well of finite diameter due to constant discharge.

Solution 9: Slug test for a finite-diameter well in a nonleaky aquifer

Assumptions:

1. A volume of water, V, is injected into, or is discharged from, the well instantaneously at $t=0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky, and flow is in radial direction only.

Differential equation:

$$\partial^2 h/\partial r^2 + (1/r)\,\partial h/\partial r = (S/T)\,\partial h/\partial t,\ r>r_w$$

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer beyond the radius of the injected well.

Boundary and initial conditions:

$$h(r_w,t)=H(t),\ t>0 \tag{1}$$

$$h(\infty,t)=0,\ \mathrm{t}>0 \tag{2}$$

$$2\pi r_w T\,\frac{\partial h(r_w,t)}{\partial r} = \pi r_c^2\,\frac{\partial H(t)}{\partial t},\ t>0 \tag{3}$$

$$h(r,0)=0,\ r>r_w \tag{4}$$

$$H(0) = H_0 = V/\pi r_c^2 \tag{5}$$

Equation 1 states that the head change in the aquifer at the face of the well is equal to that inside the well; one assumes that there is no exit loss at the well face. Equation 2 states that the head change approaches zero as distance from the discharging well approaches infinity, a condition which will be approximated if boundaries of the aquifer are sufficiently distant from the discharging well. Equation 3 states that near the well the radial flow is equal to the rate of change in volume of water inside the well. Equations 4 and 5 state that initially the head change is zero in the aquifer, and the head increase or decrease inside the well is equal to H_0.

Solution (Cooper and others, 1967):

$$h = (2H_0/\pi)\int_0^{\infty} \Big(\exp(-\beta u^2/\alpha)\big\{ J_0(ur/r_w) \cdot \big[uY_0(u) - 2\alpha Y_1(u)\big] - Y_0(ur/r_w) \cdot \big[uJ_0(u) - 2\alpha J_1(u)\big]\big\}/\Delta(u)\Big)\,du, \tag{6}$$

where

$$\alpha = r_w^2 S/r_c^2,$$

$$\beta = Tt/r_c^2,$$

and

$$\Delta(u) = \big[uJ_0(u) - 2\alpha J_1(u)\big]^2 + \big[uY_0(u) - 2\alpha Y_1(u)\big]^2.$$

J_0 and Y_0, J_1 and Y_1, are zero-order and first-order Bessel functions of the first and second kind, respectively.

The head, H, inside the well, obtained by substituting $r=r_w$ in equation (6) is

$$H/H_0 = F(\beta,\alpha),$$

where

$$F(\beta,\alpha) = (8\alpha/\pi^2)\int_0^{\infty} (\exp(-\beta u^2/\alpha)/u\Delta(u))\,du$$

and where α, β, $\Delta(u)$ are as defined previously.

Comments: Figure 9.1 is a cross section showing geometric configuration along the well bore. The volume of water injected into or discharged from the well is $\pi r_c^2 H_0$. The water-level data in the injected well, expressed as a fraction of H_0, is plotted versus time on semilogarithmic graph paper. This plot is superimposed on figure 9.2, keeping the baselines the same and sliding horizontally until a match or interpolated fit is made. A match point for β, t, and α is picked from the two graphs. Transmissivity is calculated from $T=\beta r_c^2/t$ and storage coefficient from $S = \alpha r_c^2/r_w^2$. As pointed out by Cooper, Bredehoeft, and Papadopulos (1967, p. 267), the determination of S by this method has questionable reliability because of the similar shape of the curves, whereas the determination of T is not as sensitive to choosing the correct curve. Figure 9.2 on plate 1 is plotted from data in table 9.1, which contains original material from two sources (Cooper and others, 1967; and Papadopulos and others, 1973).

Table 9.2 is a listing of a FORTRAN program by S. S. Papadopulos that evaluates $F(\beta,\alpha)$. Input to the program consists of cards coded in a specific format (readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual). Input consists of two or more cards, each containing a single value of

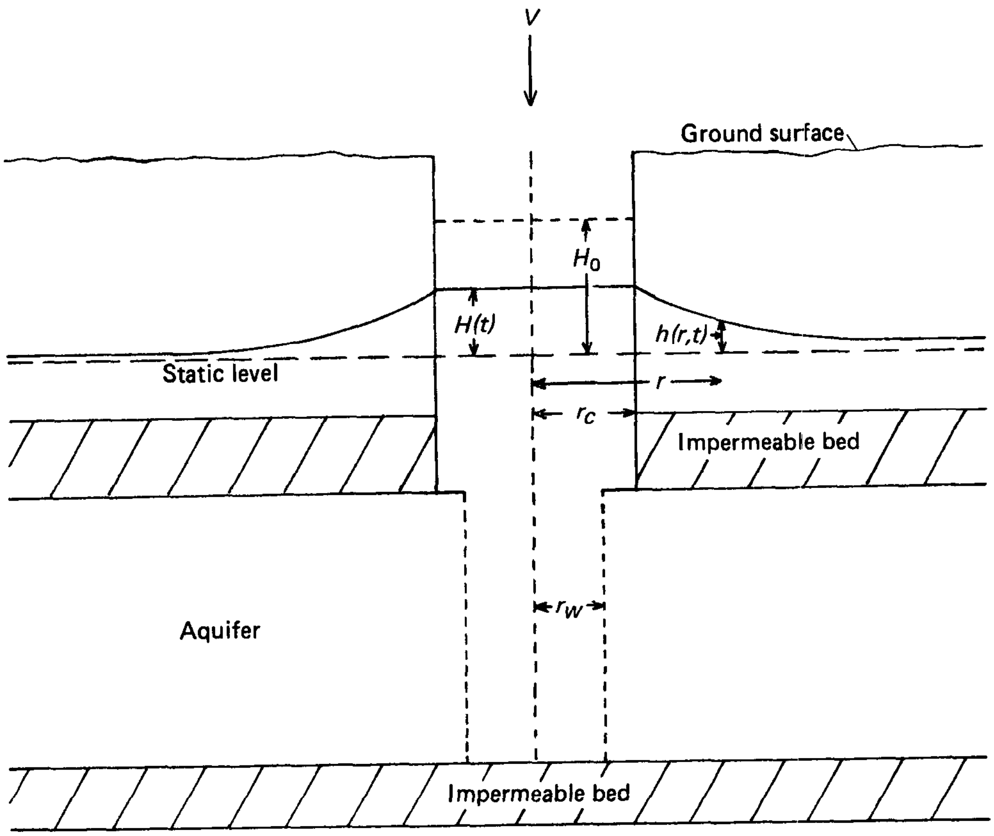

FIGURE 9.1.—Cross section through a well in which a slug of water is suddenly injected.

α coded in format F16.5. The first $\alpha \leq 0$ will signal program termination. Output from the program is shown in figure 9.3.

Solution 10: Constant discharge from a fully penetrating well in an aquifer that is anisotropic in the horizontal plane

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is anisotropic in the horizontal plane.
4. Aquifer is not leaky.
5. The transmissivity of the aquifer, T, is a two-dimensional symmetric tensor.

Differential equation:

$$T_{xx}\,\partial^2 s/\partial x^2 + 2T_{xy}\partial^2 s/\partial x\partial y + T_{yy}\partial^2 s/\partial y^2 + Q\,\delta(x)\delta(y) = S\,\partial s/\partial t.$$

This differential equation describes nonsteady flow in a homogeneous anisotropic aquifer with a constantly discharging well at $x=y=0$. The Dirac delta function is represented as $\delta(z)$ and has the following properties: $\delta(z)=0$ if $z\neq 0$ and $\int_{-\infty}^{\infty} \delta(z)dz=1$.

Boundary and initial conditions:

$$s(x,y,0)=0 \qquad (1)$$

$$s(\pm\infty,y,t)=0 \qquad (2)$$

$$s(x,\pm\infty,t)=0 \qquad (3)$$

TABLE 9.1.—*Values of* H/H_0

From Cooper, Bredehoeft, and Papadopulos, 1967

Tt/r_c^2	α	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
	1.00	0.9771	0.9920	0.9969	0.9985	0.9992
10^{-3}	2.15	.9658	.9876	.9949	.9974	.9985
	4.64	.9490	.9807	.9914	.9954	.9970
	1.00	.9238	.9693	.9853	.9915	.9942
10^{-2}	2.15	.8860	.9505	.9744	.9841	.9883
	4.64	.8293	.9187	.9545	.9701	.9781
	1.00	.7460	.8655	.9183	.9434	.9572
10^{-1}	2.15	.6289	.7782	.8538	.8935	.9167
	4.64	.4782	.6436	.7436	.8031	.8410
	1.00	.3117	.4598	.5729	.6520	.7080
10^{0}	2.15	.1665	.2597	.3543	.4364	.5038
	4.64	.07415	.1086	.1554	.2082	.2620
	7.00	.04625	.06204	.08519	.1161	.1521
	1.00	.03065	.03780	.04821	.06355	.08378
	1.40	.02092	.02414	.02844	.03492	.04426
10^{1}	2.15	.01297	.01414	.01545	.01723	.01999
	3.00	.009070	.009615	.01016	.01083	.01169
	4.64	.005711	.004919	.006111	.006319	.006554
	7.00	.003722	.003809	.003884	.003962	.004046
	1.00	.002577	.002618	.002653	.002688	.002725
10^{2}	2.15	.001179	.001187	.001194	.001201	.001208

From Papadopulos, Bredehoeft, and Cooper, 1973

Tt/r_c^2	α	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
	1	0.9994	0.9996	0.9996	0.9997	0.9997
	2	.9989	.9992	.9993	.9994	.9995
10^{-3}	4	.9980	.9985	.9987	.9989	.9991
	6	.9972	.9978	.9982	.9984	.9986
	8	.9964	.9971	.9976	.9980	.9982
	1	.9956	.9965	.9971	.9975	.9978
	2	.9919	.9934	.9944	.9952	.9958
10^{-2}	4	.9848	.9875	.9894	.9908	.9919
	6	.9782	.9819	.9846	.9866	.9881
	8	.9718	.9765	.9799	.9824	.9844
	1	.9655	.9712	.9753	.9784	.9807
	2	.9361	.9459	.9532	.9587	.9631
10^{-1}	4	.8828	.8995	.9122	.9220	.9298
	6	.8345	.8569	.8741	.8875	.8984
	8	.7901	.8173	.8383	.8550	.8686
	1	.7489	.7801	.8045	.8240	.8401
	2	.5800	.6235	.6591	.6889	.7139
	3	.4554	.5033	.5442	.5792	.6096
	4	.3613	.4093	.4517	.4891	.5222
10^{0}	5	.2893	.3351	.3768	.4146	.4487
	6	.2337	.2759	.3157	.3525	.3865
	7	.1903	.2285	.2655	.3007	.3337
	8	.1562	.1903	.2243	.2573	.2888
	9	.1292	.1594	.1902	.2208	.2505
	1	.1078	.1343	.1620	.1900	.2178
	2	.02720	.03343	.04129	.05071	.06149
	3	.01286	.01448	.01667	.01956	.02320
10^{1}	4	.008337	.008898	.009637	.01062	.01190
	5	.006209	.006470	.006789	.007192	.007709
	6	.004961	.005111	.005283	.005487	.005735
	8	.003547	.003617	.003691	.003773	.003863
	1	.002763	.002803	.002845	.002890	.002938
10^{2}	2	.001313	.001322	.001330	.001339	.001348

```
F(BETA.ALPHA) FOR ALPHA= 1.00D-01


          BETA                 H/H0

        1.00D-03             0.9769
        2.00D-03             0.9670
        4.00D-03             0.9528
        6.00D-03             0.9417
        8.00D-03             0.9322
        1.00D-02             0.9238
        2.00D-02             0.8904
        4.00D-02             0.8421
        6.00D-02             0.8048
        8.00D-02             0.7734
        1.00D-01             0.7459
        2.00D-01             0.6418
        4.00D-01             0.5095
        6.00D-01             0.4227
        8.00D-01             0.3598
        1.00D 00             0.3117
        2.00D 00             0.1786
        3.00D 00             0.1196
        4.00D 00             0.0876
        5.00D 00             0.0681
        6.00D 00             0.0553
        7.00D 00             0.0463
        8.00D 00             0.0396
        9.00D 00             0.0346
        1.00D 01             0.0306
        2.00D 01             0.0141
        3.00D 01             0.0091
        4.00D 01             0.0067
        5.00D 01             0.0053
        6.00D 01             0.0044
        7.00D 01             0.0037
        8.00D 01             0.0032
        9.00D 01             0.0029
        1.00D 02             0.0026
        2.00D 02             0.0013
        4.00D 02             0.0006
        6.00D 02             0.0004
        8.00D 02             0.0003
        1.00D 03             0.0003
```

FIGURE 9.3.—Example of output from program to compute change in water level due to sudden injection of a slug of water into a well.

Equation 1 states that, initially, drawdown is zero. Equations 2 and 3 state that the drawdown approaches zero as distance from the discharging well approaches infinity, a condition which will be approximated if boundaries of the aquifer are sufficiently distant from the discharging well.

Solution (Papadopulos, 1965, p. 23):

$$s = \left(Q/4\pi\sqrt{T_{xx}T_{yy}-T_{xy}^2}\right) W(u_{xy}), \qquad (4)$$

where

$$W(u) = \int_u^\infty (e^{-v}/v)\,dv$$

and

$$u_{xy} = (S/4t)(T_{xx}y^2 + T_{yy}x^2 - 2T_{xy}xy)/(T_{xx}T_{yy} - T_{xy}^2). \qquad (5)$$

If the coordinate axes x and y are the same as the principal axes ϵ and η (fig. 10.1) of the transmissivity tensor, the preceding equation for drawdown becomes

$$s = \left(Q/4\pi\sqrt{T_{\epsilon\epsilon}\;T_{\eta\eta}}\right) W(u_{\epsilon\eta}),$$

where

$$u_{\epsilon\eta} = (S/4t)(T_{\epsilon\epsilon}\;\eta^2 + T_{\eta\eta}\,\epsilon^2)/T_{\epsilon\epsilon}\;T_{\eta\eta}\,.$$

Comments: The method of type-curve solution as outlined by Papadopulos (1965, p. 26) requires observation of drawdown in at least three observation wells. First, choose a convenient rectangular coordinate system with the pumped well at the origin. Then, plot the observed drawdown versus t on logarithmic paper. Match these plots to the $W(u)$ type curve given in solution 1. Choose a match point of (t,s) and $(1/u_{xy}, W(u_{xy}))$ for each well and compute $T_{xx}T_{yy}-T_{xy}^2 = (QW(u_{xy})/4\pi s)^2$ for each well. Match points for all observation wells should yield approximately the same value of $(T_{xx}T_{yy}-T_{xy}^2)$. Usually they will not and judgment must be used to obtain an "average" value. Substituting this value and the three values of (x,y) in equation 5 gives three equations in three unknowns ST_{xx}, ST_{yy}, and ST_{xy}. These equations are of the form

$$y^2(ST_{xx}) + x^2(ST_{yy}) - 2xy(ST_{xy}) = 4tu_{xy}(T_{xx}T_{yy} - T_{xy}^2).$$

Solve these three equations to determine T_{xx}, T_{xy}, and T_{yy} in terms of S, and S may be determined from

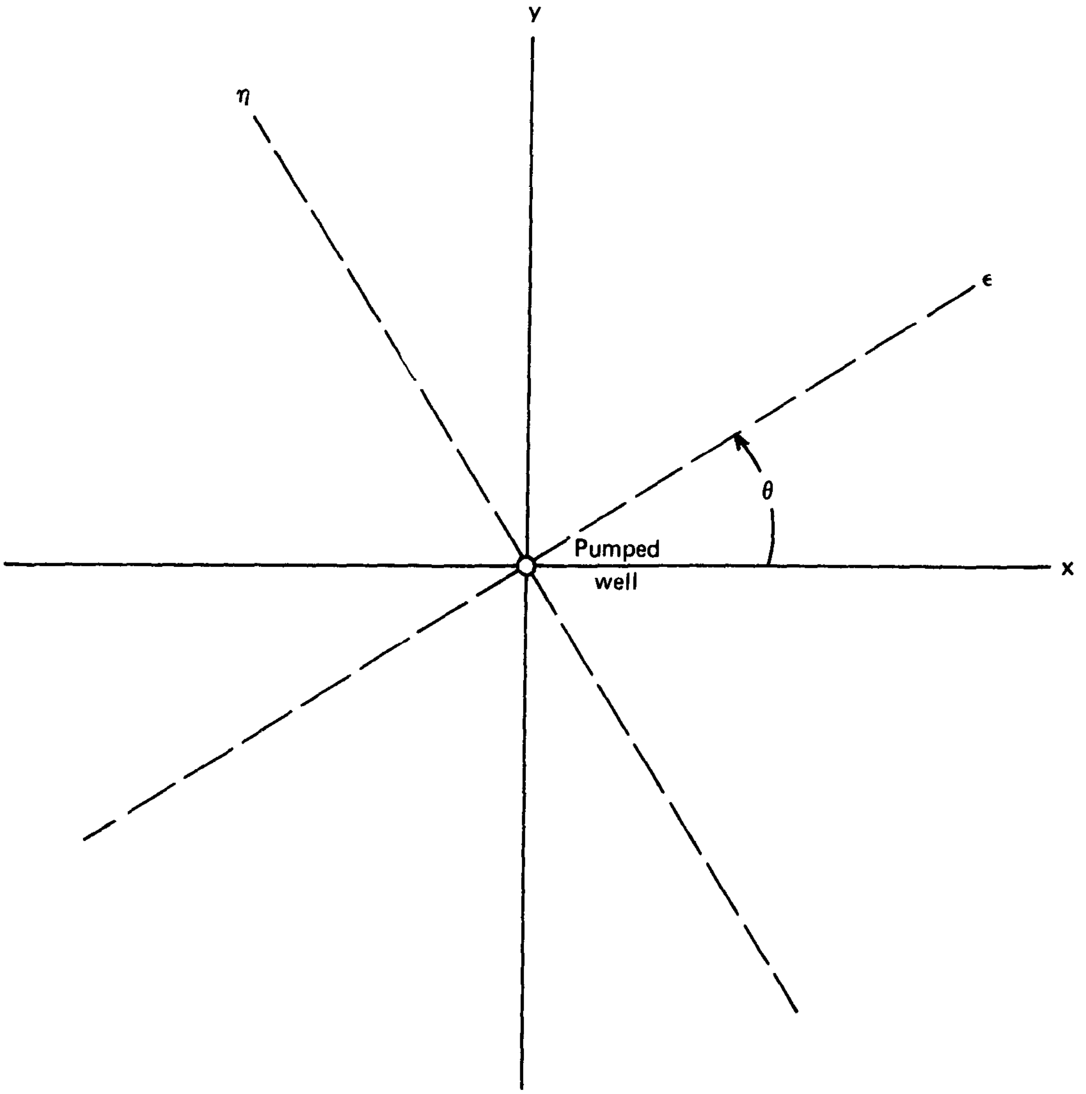

FIGURE 10.1.—Plan view showing coordinate axes.

$$S = \sqrt{(ST_{xx}ST_{yy} - (ST_{xy})^2)/(T_{xx}T_{yy} - T_{xy}^2)}.$$

Then, compute T_{xx}, T_{yy}, and T_{xy} from ST_{xx}, ST_{yy}, and ST_{xy}. $T_{\epsilon\epsilon}$, $T_{\eta\eta}$, and Θ (the angle between the x and the ϵ axis) may be calculated from the relations (Papadopulos, 1965, p. 28)

$$T_{\epsilon\epsilon} = 1/2(T_{xx} + T_{yy} + ((T_{xx} - T_{yy})^2 + 4T_{xy}^2)^{1/2})$$

$$T_{\eta\eta} = 1/2(T_{xx} + T_{yy} - ((T_{xx} - T_{yy})^2 + 4T_{xy}^2)^{1/2})$$

$$\Theta = \arctan((T_{\epsilon\epsilon} - T_{xx})/T_{xy}).$$

Solution 11: Variable discharge from a fully penetrating well in a leaky aquifer

Assumptions:

1. Well discharge changes as a specified function of time.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity (K') and thickness (b').

4. Confining bed is overlain, or underlain, by an infinite constanthead plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption will be approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{sK'}{Tb'} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

$$s(r,0)=0 \qquad (1)$$

$$s(\infty,t)=0 \qquad (2)$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q(t)}{2\pi T},\ t \geqslant 0 \qquad (3)$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is zero at large distances from the pumped well. Equation 3 states that near the pumped well the radial flow is equal to the discharge of the pumped well, which is a function of time.

Solution:

Solutions for certain discharge functions have been published by Abu-Zied and Scott (1963), and Werner (1946) for a nonleaky aquifer, and by Hantush (1964a) for both leaky and nonleaky aquifers. For arbitrary discharge functions for leaky aquifers, a solution using the convolution integral has been presented by Moench (1971, eq. 3):

$$s = (1/4\pi T)\int_0^t (Q(t')/(t-t'))$$
$$\cdot \exp\,(-A/(t-t') - (t-t')K'/Sb')dt', \qquad (4)$$

where $Q(t)$ is the discharge function of time and $A = r^2S/4T$. A numerical integration scheme is generally necessary to evaluate the above equation.

For type curves, a more useful form of equation 4 is

$$s = (Q_r/4\pi T)\int_0^t \left[Q(t')/Q_r\ (t-t')\right]$$
$$\cdot \exp\left[-A/(t-t')-(t-t')K'/Sb'\right]dt', \qquad (5)$$

or

$$s = (Q_r/4\pi T)\ SO(t), \qquad (6)$$

where $SO(t)$, read "system output function," represents the integral expression in equation 5, and Q_r is an arbitrary discharge that eliminates dimension from the integral expression. For example, Q_r could be the initial, final, or average discharge, according to the needs of the user.

Comments: Figure 11.1 is a cross section through the discharging well. This situation is the same as for solution 4, except for the varying discharge of the well. The effect of finite well radius (r_w) was investigated by Hantush (1964b, p. 4224), who concluded that for $t>25r_w^2S/T$ and $r_w/\sqrt{Tb'/K'} < 0.1$ the drawdown could be represented closely by the convolution integral.

Figure 11.2 on plate 1 shows a selected set of type curves for linear change in discharge in a nonleaky aquifer. The solution for this type of discharge function has been presented by Werner (1946, p. 706). The discharge function for figure 12.2 is $Q(t)=Q_0(1+ct)$, and the resulting drawdown is

$$s = (Q_0/4\pi T)W(u)\left\{1+ct\left[u+1-e^{-u}/W(u)\right]\right\},$$

where $W(u)$ is the well function of Theis. Substituting A/u for t in the above expression gives

$$s = (Q_0/4\pi T)\ W(u)$$
$$\cdot(1+cA\left\{1+(1/u)\left[1-e^{-u}/W(u)\right]\right\}),$$

or

$$s = (Q_0/4\pi T)\ SO(t),$$

where $SO(t)$ represents

$$W(u)(1+cA\left\{1+(1/u)\left[1-e^{-u}/W(u)\right]\right\}).$$

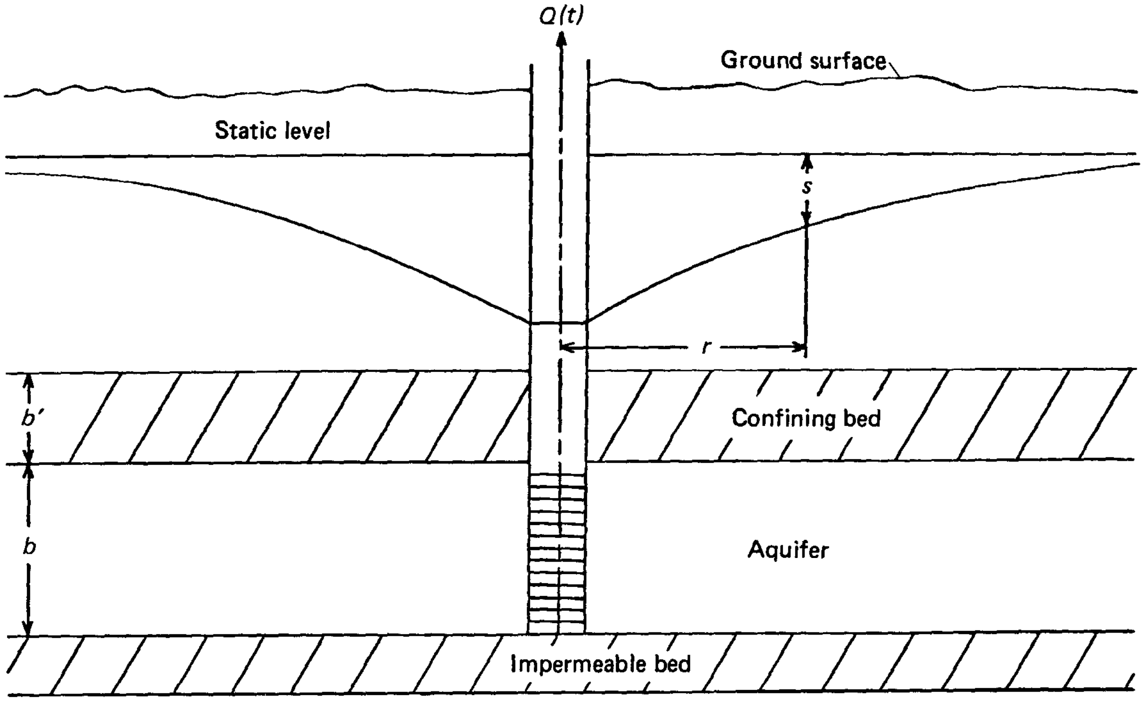

FIGURE 11.1.—Cross section through a well with variable discharge.

This substitution permits the plotting of a family of type curves, each curve specified by a value of cA.

Table 11.1 is the listing of a FORTRAN program designed to evaluate the above convolution integral for five different discharge functions. Three of these discharge functions are those devised by Hantush (1964a, p. 343, 344), who presented solutions for drawdown resulting from these functions. These three discharge functions are:

$$\text{(a)}\quad Q(t)=Q_s\left[1+\delta \exp\left(-t/t^*\right)\right],$$

$$\text{(b)}\quad Q(t)=Q_s\left[1+\delta/(1+t/t^*)\right],$$

$$\text{and}\quad \text{(c)}\quad Q(t)=Q_s\left[1+\delta/\sqrt{1+t/t^*}\right],$$

where Q_s is the ultimate steady discharge and δ and t^* are parameters defining a particular function. The first discharge function, for an exponentially decreasing discharge (case "a" of Hantush, 1964a) is virtually the same as the discharge function of Abu-Zied and Scott (1963). Besides the three functions of Hantush, the program also includes discharge as a fifth-degree polynomial of time, $Q(t)=\sum_{i=0}^{5} a_i t^i$, where the a_i are the coefficients of the polynomial, and as a piecewise linear function of time with eight segments,

$$Q(t)=a_j+b_j(t-t_{j-i})$$

for

$$t_{j-1}<t\leq t_j,\ j=1,\ 2,\ \ldots,\ 8,$$

where a_j and b_j are parameters defining the j^{th} line segment. The program uses a different, but equivalent to equation 4, expression for the convolution integral

$$s = (1/4\pi T)\int_0^t (Q(t-t')/t') \cdot \exp\left(-A/t'-t'K'/Sb'\right) dt'.$$

The program uses a sum to approximate the convolution integral. It chooses a starting value of t' that satisfies $r^2S/4Tt' + K't'/Sb' = 100$. If such a value of t' does not exist, that is, $(r^2S/4T)\,(K'/Sb')>2500$, then a value of zero is assigned for the integral value. The ending point of the interval is picked as 10 times the starting point. The integral over this interval is approximated by a trapezoidal sum using 500 subdivisions of the interval. A new interval is then constructed using the previous end point as a new starting point and a new ending point equal to 10 times the new starting point. This new interval is again evaluated by a trapezoidal sum of 500 segments. This summation procedure over intervals that are successively an order of magnitude larger continues until either $t'=t$ or $(r^2S/4Tt')$ + $(K't/Sb')$ >101. Input to this program consists of cards coded in specific formats. Readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual. Input consists of one or more groups of data, each group consisting of the following. First, one card containing the beginning time of the period of analysis in columns 1–10, coded in format E10.3; the ending time coded in columns 1311–20, in format E10.3; and a discharge index (a number from 1 through 5) coded in column 25, in format I1; and a reference discharge, QR, coded in columns 31–40, in format E10.3. The discharge index, IQ, selects a discharge function, $Q(t)$, in the following manner. If $IQ=1$, the discharge function is exponentially decreasing,

$$Q(t)=Q_s\left[1+\delta\exp(-t/t^*)\right].$$

This is case (a) of Hantush (1964a, p. 343). If $IQ=2$, the discharge function is hyperbolically decreasing,

$$Q(t)=Q_s\left[1+\delta/(1+t/t^*)\right].$$

This is case (b) of Hantush (1964a, p. 344). If $IQ=3$, the discharge function is the same as case (c) of Hantush (1964a, p. 344),

$$Q(t)=Q_s\left[1+\delta/\sqrt{1+t/t^*}\right].$$

If $IQ=4$, the discharge function is a fifth-degree polynomial of time,

$$Q(t)=\sum_{i=0}^{5} a_i t^i.$$

If $IQ=5$, the discharge function is a piecewise-linear function of time with eight or less segments,

$$Q(t)=a_j+b_j(t-t_{j-1})$$

for $\quad t_{j-1}<t\leq t_j,\ j=1,2,\ldots,8.$

The reference discharge, QR, is used to determine the form of the output from the program: If QR is coded as zero (or blank), the output shows t, s (as defined by eq. 4), and $Q(t)$. If a value greater than zero is coded for QR, the output shows $1/u$, $SO(t)$ (as defined by eq. 6), and $Q(t)/QR$.

Second, there are one or more cards containing parameters of the discharge function. If $IQ=1$, 2, or 3, then it consists of one card containing: QST, the ultimate steady discharge, coded in columns 1–10, in format E10.3; DELTA, a rate parameter, coded in columns 11–20, in format E10.3; TSTAR, a time parameter, coded in columns 21–30, in format E10.3. If $IQ=4$, it is one card containing the six polynomial coefficients. They are coded in the order $a_0, a_1, \ldots, a_5$, in columns 1–10; 11–20, . . . , 51–60 all in format E10.3. If $IQ=5$, then the program requires four cards, each card containing t_j, a_j, b_j, t_{j+1}, a_{j+1}, b_{j+1}; the four cards representing $j=1$, 3, 5, 7. The last part of each set of data consists of two or more cards containing coded values for: distance from pumped well, in columns 1–10; storage coefficient, in columns 11–20; transmissivity, in columns 21–30; and ratio of hydraulic conductivity to thickness for the confining bed, in columns 31–40, all in format E10.3. A blank card is used to signal the end of each set of data. Output from this program is shown in figure 11.3.

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Abu-Zied, M. A., and Scott, V. H., 1963, Nonsteady flow for wells with decreasing discharge: Hydraulics Div. Jour., Am. Soc. Civil Engineers Proc., v. 89, no. H-13, p. 119–132.

Cooper, H. H., Jr., 1963, Type curves for nonsteady radial flow in an infinite leaky artesian aquifer, *in* Bentall, Ray, compiler, Shortcuts and special problems in aquifer tests: U.S Geol. Survey Water-Supply Paper 1545–C, p. C48–C55.

Cooper, H. H., Jr., Bredehoeft, J. D., and Papadopulos, I. S., 1967, Response of a finite-diameter well to an instantaneous charge of water: Water Resources Research, v. 3, no. 1, p. 263–269

Dudley, W. W., Jr., 1970, Nonsteady inflow to a chamber

```
R**2*S/(4*TRANS)= 1.000E-04, K'/(S*B')= 2.500E 03, QR= 1.257E 05

1/U     1/U*10** 0           1/U*10** 1           1/U*10** 2           1/U*10** 3
       SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR
1.0    0.185  1.000E 00    0.819  1.000E 00    0.842  1.000E 00    0.842  1.000E 00
1.5    0.317  1.000E 00    0.837  1.000E 00    0.842  1.000E 00    0.842  1.000E 00
2.0    0.421  1.000E 00    0.841  1.000E 00    0.842  1.000E 00    0.842  1.000E 00
3.0    0.566  1.000E 00    0.842  1.000E 00    0.842  1.000E 00    0.842  1.000E 00
5.0    0.715  1.000E 00    0.842  1.000E 00    0.842  1.000E 00    0.842  1.000E 00
7.0    0.780  1.000E 00    0.842  1.000E 00    0.842  1.000E 00    0.842  1.000E 00

R**2*S/(4*TRANS)= 1.000E-04, K'/(S*B')= 2.500E 01, QR= 1.257E 05

1/U     1/U*10** 0           1/U*10** 1           1/U*10** 2           1/U*10** 3
       SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR
1.0    0.219  1.000E 00    1.805  1.000E 00    3.815  1.000E 00    4.829  1.000E 00
1.5    0.397  1.000E 00    2.167  1.000E 00    4.111  1.000E 00    4.849  1.000E 00
2.0    0.558  1.000E 00    2.427  1.000E 00    4.296  1.000E 00    4.853  1.000E 00
3.0    0.826  1.000E 00    2.793  1.000E 00    4.515  1.000E 00    4.854  1.000E 00
5.0    1.216  1.000E 00    3.244  1.000E 00    4.708  1.000E 00    4.854  1.000E 00
7.0    1.495  1.000E 00    3.530  1.000E 00    4.785  1.000E 00    4.854  1.000E 00

R**2*S/(4*TRANS)= 1.000E-04, K'/(S*B')= 2.500E-01, QR= 1.257E 05

1/U     1/U*10** 0           1/U*10** 1           1/U*10** 2           1/U*10** 3
       SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR
1.0    0.219  1.000E 00    1.823  1.000E 00    4.036  1.000E 00    6.307  1.000E 00
1.5    0.398  1.000E 00    2.196  1.000E 00    4.437  1.000E 00    6.700  1.000E 00
2.0    0.560  1.000E 00    2.468  1.000E 00    4.721  1.000E 00    6.975  1.000E 00
3.0    0.829  1.000E 00    2.857  1.000E 00    5.123  1.000E 00    7.356  1.000E 00
5.0    1.223  1.000E 00    3.354  1.000E 00    5.627  1.000E 00    7.820  1.000E 00
7.0    1.507  1.000E 00    3.684  1.000E 00    5.958  1.000E 00    8.110  1.000E 00

R**2*S/(4*TRANS)= 1.000E-04, K'/(S*B')= 2.500E-03, QR= 1.257E 05

1/U     1/U*10** 0           1/U*10** 1           1/U*10** 2           1/U*10** 3
       SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR     SO(T)    Q(T)/QR
1.0    0.219  1.000E 00    1.823  1.000E 00    4.038  1.000E 00    6.332  1.000E 00
1.5    0.398  1.000E 00    2.197  1.000E 00    4.440  1.000E 00    6.737  1.000E 00
2.0    0.560  1.000E 00    2.468  1.000E 00    4.726  1.000E 00    7.024  1.000E 00
3.0    0.829  1.000E 00    2.857  1.000E 00    5.130  1.000E 00    7.429  1.000E 00
5.0    1.223  1.000E 00    3.355  1.000E 00    5.639  1.000E 00    7.939  1.000E 00
7.0    1.507  1.000E 00    3.686  1.000E 00    5.975  1.000E 00    8.275  1.000E 00
```

FIGURE 11.3.—Example of output from program to compute the convolution integral for a leaky aquifer.

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SUPPLEMENTAL DATA

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*

```
C     ******************************************************************
C
C     PURPOSE
C        TO COMPUTE TYPE CURVE FUNCTION VALUES FOR PARTIAL PENETRATION
C        IN A NONLEAKY AQUIFER USING EQUATIONS 1 AND 9A OF HANTUSH,M.S.,
C        1961,DRAWDOWN AROUND A PARTIALLY PENETRATING WELL; HYDRAULIC
C        DIV. JOUR., AM. SOC. CIVIL ENGINEERS PROC., P. 83-98.
C     INPUT DATA
C        1 CARD  -  FORMAT (3F5.1,I5,2E10.4)
C              B = AQUIFER THICKNESS
C              L = DEPTH, BELOW TOP OF AQUIFER, TO BOTTOM OF PUMPING
C              WELL SCREEN
C              D = DEPTH, BELOW TOP OF AQUIFER, TO TOP OF PUMPING WELL
C              SCREEN
C              NUM = NUMBER OF OBSERVATION WELLS OR PIEZOMETERS TIMES
C              NUMBER OF VALUES OF KZ/KR.
C              SMALL = SMALLEST VALUE OF 1/U FOR WHICH COMPUTATION IS
C              DESIRED
C              LARGE = LARGEST  VALUE OF 1/U FOR WHICH COMPUTATION IS
C              DESIRED
C        NUM CARDS (ONE FOR EACH OBS. WELL OR PIEZOMETER AND FOR EACH  1
C              VALUE OF R*SQRT(KZ/KR), - FORMAT (3F5.1)
C              R = RADIAL DISTANCE FROM PUMPED WELL TIMES SQRT(KZ/KR).
C              LPRIME = DEPTH, BELOW TOP OF AQUIFER, TO BOTTOM OF OBS.
C              WELL SCREEN (ZERO FOR PIEZOMETER)
C              DPRIME = DEPTH, BELOW TOP OF AQUIFER, TO TOP OF OBS. WELL
C              SCREEN (TOTAL DEPTH FOR PIEZOMETER)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT,L,F,EXPI
C
C     ******************************************************************
      REAL*8 U
      REAL*4 L,LB,LPB,LPRIME,LARGE
      DIMENSION ARRAY(13,12), IARG(12), ARG(13), A(12), C(12)
      DATA ARG/1.,1.2,1.5,2.,2.5,3.,3.5,4.,5.,6.,7.,8.,9./
      DATA A/12*'  N*'/,C/12*'10**'/
      IRD=5
      IPT=6
      READ (IRD,6) B,L,D,NUM,SMALL,LARGE
      LB=L/B
      DB=D/B
      IBEGIN=ALOG10(SMALL)
      IEND=ALOG10(LARGE)+1.
      JLIMIT=IEND-IBEGIN
      IF (JLIMIT.GT.12) JLIMIT=12
      DO 5 K=1,NUM
      READ (IRD,6) R,LPRIME,DPRIME
      RB=R/B
      LPB=LPRIME/B
      DPB=DPRIME/B
      DO 1 I=1,13
      ARGI=ARG(I)
      DO 1 J=1,JLIMIT
      IARG(J)=IBEGIN+J-1
      Y=ARGI*10.**(IBEGIN+J-1)
      U=1./Y
      X=U
      CALL EXPI(X,WU,DUMMY)
    1 ARRAY(I,J)=WU+F(U,RB,LB,DB,LPB,DPB)
      IF (LPB-0.) 2,2,3
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
    2 WRITE (IPT,7) DPB,RB,LB,DB
      GO TO 4
    3 WRITE (IPT,8) LPB,DPB,RB,LB,DB
    4 WRITE (IPT,9) (A(I),C(I),IARG(I),I=1,JLIMIT)
      DO 5 I=1,13
      WRITE (IPT,10) ARG(I),(ARRAY(I,J),J=1,JLIMIT)
    5 CONTINUE
      STOP
C
C
    6 FORMAT (3F5.1,I5,2E10.4)
    7 FORMAT ('1','W(U)+F(U,R/B,L/B,D/B,Z/B), Z/B=',F5.2,', SQRT(KZ/KR)*
     1R/B=',F5.2,', L/B=',F5.2,', D/B=',F5.2,', U=1/N')
    8 FORMAT ('1','W(U)+F(U,R/B,L/B,D/B,L''/B,D''/B), L''/B=',F5.2,', D'
     1'/B=',F5.2,', SQRT(KZ/KR)*R/B=',F5.2,', L/B=',F5.2,', D/B=',F5.2,'
     2, U=1/N')
    9 FORMAT ('0',2X,'N',1X,12(2A4,I2))
   10 FORMAT ((' ',F4.1,12(F9.4,1X)))
      END
      REAL FUNCTION F*4(U,RB,LB,DB,LPB,DPB)
C     ******************************************************************
C
C     FUNCTION F
C
C     PURPOSE
C        TO COMPUTE DEPARTURES FROM THEIS CURVE CAUSED BY PARTIAL
C        PENETRATION OF PUMPED WELL.
C     USAGE
C        F(U,RB,LB,DB,LPB,DPB)
C     DESCRIPTION OF PARAMETERS
C        ALL REAL, U DOUBLE PRECISION
C        U = R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C              COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        RB = R/B ( RADIAL DISTANCE / AQUIFER THICKNESS )
C        LB = L/B ( FRACTION OF AQUIFER PENETRATED BY PUMPED WELL)
C        DB = D/B ( FRACTION OF AQUIFER ABOVE PUMPED WELL SCREEN)
C        LPB = L'/B (FRACTION OF AQUIFER PENETRATED BY OBS. WELL, ZERO
C                FOR PIEZOMETER)
C        DPB = D'/B (FRACTION OF AQUIFER ABOVE OBS. WELL SCREEN, TOTAL
C                DEPTH FOR PIEZOMETER)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT,L
C     METHOD
C        SUMS THE SERIES THROUGH N*PI*R/B EQ 20
C
C     ******************************************************************
      REAL*8 U,V
      REAL*4 L,N,LB,LPB
      SUM=0.
      N=0.
      PIRB=3.141593*RB
      PILB=3.141593*LB
      PIDB=3.141593*DB
      IF (LPB-0.) 1,1,4
C     CHECKS FOR WELL OR PIEZOMETER
    1 PIZB=3.141593*DPB
    2 N=N+1.
      V=N*PIRB/2.
      IF (V.GT.10.) GO TO 3
C     TRUNCATES SERIES WHEN V>10
      X=L(U,V)/N
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
      SUM=SUM+(SIN(N*PILB)-SIN(N*PIDB))*COS(N*PIZB)*X
      GO TO 2
    3 F=.6366198*SUM/(LB-DB)
      GO TO 7
    4 PILPB=3.141593*LPB
      PIDPB=3.141593*DPB
    5 N=N+1
      V=N*PIRB/2.
      IF (V.GT.10.) GO TO 6
C     TRUNCATES SERIES WHEN V>10
      X=L(U,V)/N
      SUM=SUM+(SIN(N*PILB)-SIN(N*PIDB))*(SIN(N*PILPB)-SIN(N*PIDPB))*X/N
      GO TO 5
    6 F=.2026424*SUM/((LB-DB)*(LPB-DPB))
    7 RETURN
      END

      REAL FUNCTION L*4(U,V)
C     ******************************************************************
C
C     FUNCTION L
C
C     PURPOSE
C        TO COMPUTE THE INTEGRAL( EXP(-Y-V**2/Y)/Y) SUMMED OVER Y FROM
C        U TO INFINITY(WELL FUNCTION FOR LEAKY AQUIFERS).
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U - R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V - R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT
C     METHOD
C        IN THE FOLLOWING F=EXP(-Y-V**2/Y)/Y
C      (1) U>=1, USES A GAUSSIAN-LAGUERRE QUADRATURE FORMULA TO
C        EVALUATE INTEGRAL(F) FROM U TO INF.
C      (2) V**2<U<1, USES THE G-L QUADRATURE TO EVALUATE INTEGRAL(F)
C        FROM ONE TO INF AND A SERIES EXPANSION TO EVALUATE INTEGRAL(F)
C        FROM U TO ONE.
C      (3) U<1, U<=V**2, USES THE REPRESENTATION INTEGRAL(F) FROM U
C        TO INF. = 2*K0(2*V)-INTEGRAL(F) FROM V**2/U TO INF.
C        EVALUATES THE ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND
C        KIND WITH IBM SUBROUTINE, EVALUATES INTEGRAL BY G-L QUAD.
C
C     ******************************************************************
      EXTERNAL FCT
      REAL*8 U,V,Z,F,VV,SERIES
      COMMON /C1/ VV,Z
      VV=V
      IF (U-1.) 1,2,2
C     CHECKS IF U<1
    1 Z=V*V/U
      IF (Z-1.) 3,4,4
C     CHECKS IF V**2/U < 1
    2 Z=U
      CALL DQL12(FCT,F)
      L=F
C     INTEGRAL U TO INF. EVALUATED BY GAUSS-LAGUERRE QUADRATURE
      GO TO 5
    3 Z=1.
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
      CALL DQL12(FCT,F)
      L=F+SERIES(U,V)
C     INTEGRAL 1 TO INF. BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP.
      GO TO 5
    4 TWOV=2.*V
      CALL BESK(TWOV,0,BK,IER)
      CALL DQL12(FCT,F)
      L=2.*BK-F
C     2K0(2V)-INTEGRAL V**2/U TO INF.
    5 RETURN
      END

      REAL FUNCTION SERIES*8(U,V)
C     ******************************************************************
C
C     FUNCTION SERIES
C
C     PURPOSE
C        TO EVALUATE S(1)-S(U), WHERE S IS A SERIES EXPANSION OF
C        INTEGRAL(EXP(-Y-V**2/Y)DY/Y) GIVEN BY: S= SUM, M=0 TO INFINITY,
C        (F(M)*SUM, N=0 TO INF.,(V**(2*N)/((N!)*(M+N)!)) WHERE F(M)=
C        LOG(U) IF M=0 AND = ((-1)**M/M)*(U**M-(V**2/U)**M) IF M>0.
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U - R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V - R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE
C     METHOD
C        SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7/N AND FOR OUTER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7
C
C     ******************************************************************
      REAL*8 DLOG,DABS,S(2),VUM,UU
      REAL*8 TEST,U,UM,EM,EN,SUM1,SUM,SIGN,V,VSQ,VSQU,RMUL,TERM,TERM1
      TEST=5.D-07
      VSQ=V*V
      UU=U
      DO 6 I=1,2
C     EVALUATES SERIES FOR LOWER LIMIT = U AND UPPER LIMIT = 1
      IF (I.EQ.2) U=1.
      UM=1.
      EM=-1.
      SUM1=0.
      SIGN=-1.
      VUM=1.
      VSQU=VSQ/U
    1 EM=EM+1.
      IF (EM-.1) 2,3,3
C     CHECKS FOR M=0
    2 RMUL=DLOG(U)
      TERM1=1.
      GO TO 4
    3 UM=UM*U
      IF (VUM.LT.1.D-30) VUM=0.
      VUM=VUM*VSQU
      RMUL=(UM-VUM)/EM
      TERM1=TERM1/EM
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
    4 SIGN=-SIGN
      SUM=TERM1
      TERM=TERM1
      EN=0.
    5 EN=EN+1.
      TERM=TERM*VSQ/(EN*(EN+EM))
      SUM=SUM+TERM
      IF (TEST.LE.DABS(RMUL*EN*TERM)) GO TO 5
C     TRUNCATES INNER SERIES IF OUTER TERM*N*INNER TERM < 5.E-7
      SUM1=SUM1+SIGN*RMUL*SUM
      IF (EM.LT..1) GO TO 1
      IF (TEST.LE.DABS(RMUL*SUM)) GO TO 1
C     TRUNCATES OUTER SERIES IF OUTER TERM*INNER SUM < 5.E-7
    6 S(I)=SUM1
      U=UU
      SERIES=S(2)-S(1)
      RETURN
      END
      REAL FUNCTION FCT*8(X)
C     ******************************************************************
C
C     FUNCTION FCT
C
C     PURPOSE
C        TO COMPUTE FCT(X)=EXP(-Z-V**2/(X+Z))/(X+Z)
C     DESCRIPTION OF PARAMETERS
C        X - THE DOUBLE PRECISION VALUE OF X FOR WHICH FCT IS COMPUTED
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE
C     METHOD
C        FORTRAN EVALUATION OF FUNCTION
C
C     ******************************************************************
      REAL*8 X,V,Z,P,DEXP
      COMMON /C1/ V,Z
      IF (X) 1,2,2
    1 FCT=0.
      GO TO 4
    2 P=Z+V**2/(X+Z)
      IF (P-5.D1) 3,3,1
    3 FCT=DEXP(-P)/(X+Z)
    4 RETURN
      END
      SUBROUTINE DQL12(FCT,Y)
C
C     ..................................................................
C
C        SUBROUTINE DQL12
C
C        PURPOSE
C           TO COMPUTE INTEGRAL(EXP(-X)*FCT(X), SUMMED OVER X
C                               FROM 0 TO INFINITY).
C
C        USAGE
C           CALL DQL12 (FCT,Y)
C           PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT
C
C        DESCRIPTION OF PARAMETERS
C           FCT    - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION
C                    SUBPROGRAM USED.
C           Y      - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
C
C        REMARKS
C           NONE
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)
C           MUST BE FURNISHED BY THE USER,
C
C        METHOD
C           EVALUATION IS DONE BY MEANS OF 12-POINT GAUSSIAN-LAGUERRE
C           QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY,
C           WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23,
C           FOR REFERENCE, SEE
C           SHAO/CHEN/FRANK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF
C           CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED
C           GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT
C           TR00,1100 (MARCH 1964), PP,24-25,
C
C     ..................................................................
C
C
C
      DOUBLE PRECISION X,Y,FCT
C
      X=,3709912104446692 D2
      Y=,8148077467426240 D-15*FCT(X)
      X=,2848796725098400 D2
      Y=Y+,3061601635035021 D-11*FCT(X)
      X=,2215109037939701 D2
      Y=Y+,1342391030515004 D-8*FCT(X)
      X=,1711685518746226 D2
      Y=Y+,1668493876540910 D-6*FCT(X)
      X=,1300605499330635 D2
      Y=Y+,8365055856819800 D-5*FCT(X)
      X=,9621316842456867 D1
      Y=Y+,2032315926629994 D-3*FCT(X)
      X=,6844525453115177 D1
      Y=Y+,2663973541865316 D-2*FCT(X)
      X=,4599227639418348 D1
      Y=Y+,2010238115463410 D-1*FCT(X)
      X=,2833751337743507 D1
      Y=Y+,9044922221168090 D-1*FCT(X)
      X=,1512610269776419 D1
      Y=Y+,2440820113198776 D0*FCT(X)
      X=,6117574845151307 D0
      Y=Y+,3777592758731380 D0*FCT(X)
      X=,1157221173580207 D0
      Y=Y+,2647313710554432 D0*FCT(X)
      RETURN
      END
      SUBROUTINE BESK(X,N,BK,IER)
C
C     ..................................................................
C
C        SUBROUTINE BESK
C
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESK(X,N,BK,IER)
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
C                                                                       BESK 100
C        DESCRIPTION OF PARAMETERS                                      BESK 110
C           X   =THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED          BESK 120
C           N   =THE ORDER OF THE K BESSEL FUNCTION DESIRED             BESK 130
C           BK  =THE RESULTANT K BESSEL FUNCTION                        BESK 140
C           IER=RESULTANT ERROR CODE WHERE                              BESK 150
C              IER=0  NO ERROR                                          BESK 160
C              IER=1  N IS NEGATIVE                                     BESK 170
C              IER=2  X IS ZERO OR NEGATIVE                             BESK 180
C              IER=3  X .GT. 170, MACHINE RANGE EXCEEDED                BESK 190
C              IER=4  BK .GT. 10**70                                    BESK 200
C                                                                       BESK 210
C        REMARKS                                                        BESK 220
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                     BESK 230
C                                                                       BESK 240
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  BESK 250
C           NONE                                                        BESK 260
C                                                                       BESK 270
C        METHOD                                                         BESK 280
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING  BESK 290
C           SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION BESK 300
C           USING RECURRENCE RELATION,                                  BESK 310
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE  BESK 320
C           AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS  BESK 330
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED    BESK 340
C           FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,   BESK 350
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE   BESK 360
C           UNIVERSITY PRESS, 1958, P. 62                               BESK 370
C                                                                       BESK 380
C     ..................................................................BESK 390
C                                                                       BESK 400
      DIMENSION T(12)                                                   BESK 420
      BK=.0                                                             BESK 430
      IF(N)10,11,11                                                     BESK 440
   10 IER=1                                                             BESK 450
      RETURN                                                            BESK 460
   11 IF(X)12,12,20                                                     BESK 470
   12 IER=2                                                             BESK 480
      RETURN                                                            BESK 490
   20 IF(X-170.0)22,22,21                                               BESK 500
   21 IER=3                                                             BESK 510
      RETURN                                                            BESK 520
   22 IER=0                                                             BESK 530
      IF(X-1.)36,36,25                                                  BESK 540
   25 A=EXP(-X)                                                         BESK 550
      B=1./X                                                            BESK 560
      C=SQRT(B)                                                         BESK 570
      T(1)=B                                                            BESK 580
      DO 26 L=2,12                                                      BESK 590
   26 T(L)=T(L-1)*B                                                     BESK 600
      IF(N-1)27,29,27                                                   BESK 610
C                                                                       BESK 620
C     COMPUTE K0 USING POLYNOMIAL APPROXIMATION                         BESK 630
C                                                                       BESK 640
   27 G0=A*(1.2533141-.1566642*T(1)+.08811128*T(2)-.09139095*T(3)       BESK 650
     2+.1344596*T(4)-.2299850*T(5)+.3792410*T(6)-.5247277*T(7)          BESK 660
     3+.5575368*T(8)-.4262633*T(9)+.2184518*T(10)-.06680977*T(11)       BESK 670
     4+.009189383*T(12))*C                                              BESK 680
      IF(N)20,28,29                                                     BESK 690
   28 BK=G0                                                             BESK 700
      RETURN                                                            BESK 710
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
C                                                                       BESK 720
C     COMPUTE K1 USING POLYNOMIAL APPROXIMATION                         BESK 730
C                                                                       BESK 740
   29 G1=A*(1.2533141+.4699927*T(1)-.1468583*T(2)+.1280427*T(3)         BESK 750
     2-.1736432*T(4)+.2847618*T(5)-.4594342*T(6)+.6283381*T(7)          BESK 760
     3-.6632295*T(8)+.5050239*T(9)-.2581304*T(10)+.07880001*T(11)       BESK 770
     4-.01082418*T(12))*C                                               BESK 780
      IF(N-1)20,30,31                                                   BESK 790
   30 BK=G1                                                             BESK 800
      RETURN                                                            BESK 810
C                                                                       BESK 820
C     FROM K0,K1 COMPUTE KN USING RECURRENCE RELATION                   BESK 830
C                                                                       BESK 840
   31 DO 35 J=2,N                                                       BESK 850
      GJ=2.*(FLOAT(J)-1.)*G1/X+G0                                       BESK 860
      IF(GJ-1.0E70)33,33,32                                             BESK 870
   32 IER=4                                                             BESK 880
      GO TO 34                                                          BESK 890
   33 G0=G1                                                             BESK 900
   35 G1=GJ                                                             BESK 910
   34 BK=GJ                                                             BESK 920
      RETURN                                                            BESK 930
   36 B=X/2.                                                            BESK 940
      A=.5772157+ALOG(B)                                                BESK 950
      C=B*B                                                             BESK 960
      IF(N-1)37,43,37                                                   BESK 970
C                                                                       BESK 980
C     COMPUTE K0 USING SERIES EXPANSION                                 BESK 990
C                                                                       BESK1000
   37 G0=-A                                                             BESK1010
      X2J=1.                                                            BESK1020
      FACT=1.                                                           BESK1030
      HJ=.0                                                             BESK1040
      DO 40 J=1,6                                                       BESK1050
      RJ=1./FLOAT(J)                                                    BESK1060
      IF(X2J.LT.1.E-40) X2J=0.                                          BESK1061
C     PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW   BESK1062
C     PROBLEM ON WATFOR COMPILER                                        BESK1063
      X2J=X2J*C                                                         BESK1070
      FACT=FACT*RJ*RJ                                                   BESK1080
      HJ=HJ+RJ                                                          BESK1090
   40 G0=G0+X2J*FACT*(HJ-A)                                             BESK1100
      IF(N)43,42,43                                                     BESK1110
   42 BK=G0                                                             BESK1120
      RETURN                                                            BESK1130
C                                                                       BESK1140
C     COMPUTE K1 USING SERIES EXPANSION                                 BESK1150
C                                                                       BESK1160
   43 X2J=B                                                             BESK1170
      FACT=1.                                                           BESK1180
      HJ=1.                                                             BESK1190
      G1=1./X+X2J*(.5+A-HJ)                                             BESK1200
      DO 50 J=2,8                                                       BESK1210
      X2J=X2J*C                                                         BESK1220
      RJ=1./FLOAT(J)                                                    BESK1230
      FACT=FACT*RJ*RJ                                                   BESK1240
      HJ=HJ+RJ                                                          BESK1250
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))                               BESK1260
      IF(N-1)31,52,31                                                   BESK1270
   52 BK=G1                                                             BESK1280
      RETURN                                                            BESK1290
      END                                                               BESK130-
```

TABLE 2.1.—*Listing of program for partial penetration in a nonleaky artesian aquifer*—Continued

```
      SUBROUTINE EXPI(X,RES,AUX)
C
C     ..................................................................
C
C        SUBROUTINE EXPI
C
C        PURPOSE
C           COMPUTES THE EXPONENTIAL INTEGRAL -EI(-X)
C
C        USAGE
C           CALL EXPI(X,RES)
C
C        DESCRIPTION OF PARAMETERS
C           X      - ARGUMENT OF EXPONENTIAL INTEGRAL
C           RES    - RESULT VALUE
C           AUX    - RESULTANT AUXILIARY VALUE
C
C        REMARKS
C           X GT 170 (X LT -174) MAY CAUSE UNDERFLOW (OVERFLOW)
C           WITH THE EXPONENTIAL FUNCTION
C           FOR X = 0 THE RESULT VALUE IS SET TO -1,E75
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           DEFINITION
C           RES=INTEGRAL(EXP(-T)/T, SUMMED OVER T FROM X TO INFINITY).
C           EVALUATION
C           THREE DIFFERENT RATIONAL APPROXIMATIONS ARE USED IN THE
C           RANGES 1 LE X, X LE -9 AND -9 LT X LE -3 RESPECTIVELY,
C           A POLYNOMIAL APPROXIMATION IS USED IN -3 LT X LT 1.
C
C     ..................................................................
C
      IF(X-1,)2,1,1
    1 Y=1,/X
      AUX=1,-Y*(((Y+3,377358E0)*Y+2,052156E0)*Y+2,709479E-1)/((((Y*
     11,072553E0+5,716943E0)*Y+6,945239E0)*Y+2,593888E0)*Y+2,709496E-1)
      RES=AUX*Y*EXP(-X)
      RETURN
    2 IF(X+3,)6,6,3
    3 AUX=(((((((7,122452E-7*X-1,766345E-6)*X+2,928433E-5)*X-2,335379E-4
     1)*X+1,664156E-3)*X-1,041576E-2)*X+5,555682E-2)*X-2,500001E-1)*X
     2+9,999999E-1
      RES=-1,E75
      IF(X)4,5,4
    4 RES=X*AUX-ALOG(ABS(X))-5,772157E-1
    5 RETURN
    6 IF(X+9,)8,8,7
    7 AUX=1,-((((5,176245E-2*X+3,061037E0)*X+3,243665E1)*X+2,244234E2)*X
     1+2,486697E2)/((((X+3,995161E0)*X+3,893944E1)*X+2,263818E1)*X
     2+1,807837E2)
      GOTO 9
    8 Y=9,/X
      AUX=1,-Y*(((Y+7,659824E-1)*Y-7,271015E-1)*Y-1,080693E0)/((((Y
     1*2,518750E0+1,122927E1)*Y+5,921405E0)*Y-8,666702E0)*Y-9,724216E0)
    9 RES=AUX*EXP(-X)/X
      RETURN
      END
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*

```
C     ******************************************************************
C
C     PURPOSE
C        TO COMPUTE A TABLE OF VALUES OF THE LEAKY AQUIFER WELL
C        FUNCTION - W(U,R/B) - HANTUSH,M.S., AND JACOB,C.E., 1955,
C        NON-STEADY RADIAL FLOW IN AN INFINITE LEAKY AQUIFER: AM.
C        GEOPHYS. UNION TRANS., V. 36, NO. 1, P. 95-100.
C     INPUT DATA
C        1 CARD - FORMAT(2E10.5)
C              USMALL - SMALLEST VALUE OF 1/U FOR WHICH COMPUTATION IS
C              DESIRED.
C              ULARGE - LARGEST  VALUE OF 1/U FOR WHICH COMPUTATION IS
C              DESIRED.
C        2 CARDS - FORMAT(8E10.5)
C              BDAT - 12 VALUES OF R/B FOR TABLE.
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        L,SERIES,FCT,BESK,DQL12
C
C     ******************************************************************
      REAL*4 L
      REAL*8 U,V
      DIMENSION ARRAY(73,12), Y(73), BDAT(12), YNUM(6)
      DATA YNUM/1.,1.5,2.,3.,5.,7./
      IRD=5
      IPT=6
      READ (IRD,6) USMALL,ULARGE
      READ (IRD,6) BDAT
      IBEGIN=ALOG10(USMALL)
      IEND=ALOG10(ULARGE)+.99999
      ILIMIT=(IEND-IBEGIN)*6+1
      IF (ILIMIT.GT.73) ILIMIT=73
      DO 1 I=1,12
      IF (BDAT(I).EQ.0.) GO TO 2
    1 CONTINUE
      NB=12
      GO TO 3
    2 NB=I-1
    3 II=0
      DO 4 I=1,ILIMIT
      II=II+1
      IF (II.GT.6) II=1
      IEXP=IBEGIN+(I-1)/6
      Y(I)=YNUM(II)*10.**IEXP
      U=1./Y(I)
      DO 4 J=1,NB
      V=BDAT(J)/2.
    4 ARRAY(I,J)=L(U,V)
      WRITE (IPT,7) (BDAT(I),I=1,NB)
      DO 5 I=1,ILIMIT
    5 WRITE (IPT,8) Y(I),(ARRAY(I,J),J=1,NB)
      STOP
C
C
    6 FORMAT (8E10.5)
    7 FORMAT ('1','W(U,R/B)'/'0',10X,'| R/B'/' ',6X,'1/U |',12E10.2)
    8 FORMAT (' ',E10.3,12F10.4)
      END
      REAL FUNCTION L*4(U,V)
C     ******************************************************************
C
C     FUNCTION L
C
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
C     PURPOSE
C        TO COMPUTE THE INTEGRAL( EXP(-Y-V**2/Y)/Y) SUMMED OVER Y FROM
C        U TO INFINITY(WELL FUNCTION FOR LEAKY AQUIFERS).
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U - R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V - R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT
C     METHOD
C        IN THE FOLLOWING F=EXP(-Y-V**2/Y)/Y
C      (1) U>=1, USES A GAUSSIAN-LAGUERRE QUADRATURE FORMULA TO
C        EVALUATE INTEGRAL(F) FROM U TO INF.
C      (2) V**2<U<1, USES THE G-L QUADRATURE TO EVALUATE INTEGRAL(F)
C        FROM ONE TO INF AND A SERIES EXPANSION TO EVALUATE INTEGRAL(F)
C        FROM U TO ONE.
C      (3) U<1, U<=V**2, USES THE REPRESENTATION INTEGRAL(F) FROM U
C        TO INF. = 2*K0(2*V)-INTEGRAL(F) FROM V**2/U TO INF.
C        EVALUATES THE ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND
C        KIND WITH IBM SUBROUTINE, EVALUATES INTEGRAL BY G-L QUAD.
C
C     ******************************************************************
      EXTERNAL FCT
      REAL*8 U,V,Z,F,VV,SERIES
      COMMON /C1/ VV,Z
      VV=V
      IF (U-1.) 1,2,2
C     CHECKS IF U<1
    1 Z=V*V/U
      IF (Z-1.) 3,4,4
C     CHECKS IF V**2/U < 1
    2 Z=U
      CALL DQL12(FCT,F)
      L=F
C     INTEGRAL U TO INF. EVALUATED BY GAUSS-LAGUERRE QUADRATURE
      GO TO 5
    3 Z=1.
      CALL DQL12(FCT,F)
      L=F+SERIES(U,V)
C     INTEGRAL 1 TO INF. BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP.
      GO TO 5
    4 TWOV=2.*V
      CALL BESK(TWOV,0,BK,IER)
      CALL DQL12(FCT,F)
      L=2.*BK-F
C     2K0(2V)-INTEGRAL V**2/U TO INF.
    5 RETURN
      END
      REAL FUNCTION SERIES*8(U,V)
C     *******************************************************************
C
C     FUNCTION SERIES
C
C     PURPOSE
C        TO EVALUATE S(1)-S(U), WHERE S IS A SERIES EXPANSION OF
C        INTEGRAL(EXP(-Y-V**2/Y)DY/Y) GIVEN BY: S= SUM, M=0 TO INFINITY,
C        (F(M)*SUM, N=0 TO INF.,(V**(2*N)/((N!)*(M+N)!)) WHERE F(M)=
C        LOG(U) IF M=0 AND = ((-1)**M/M)*(U**M-(V**2/U)**M) IF M>0.
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U - R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V - R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE
C     METHOD
C        SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7/N AND FOR OUTER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7
C
C     *****************************************************************
      REAL*8 DLOG,DABS,S(2),VUM,UU
      REAL*8 TEST,U,UM,EM,EN,SUM1,SUM,SIGN,V,VSQ,VSQU,RMUL,TERM,TERM1
      TEST=5.D-07
      VSQ=V*V
      UU=U
      DO 6 I=1,2
C     EVALUATES SERIES FOR LOWER LIMIT = U AND UPPER LIMIT = 1
      IF (I.EQ.2) U=1.
      UM=1.
      EM=-1.
      SUM1=0.
      SIGN=-1.
      VUM=1.
      VSQU=VSQ/U
    1 EM=EM+1.
      IF (EM-.1) 2,3,3
C     CHECKS FOR M=0
    2 RMUL=DLOG(U)
      TERM1=1.
      GO TO 4
    3 UM=UM*U
      IF (VUM.LT.1.D-30) VUM=0.
      VUM=VUM*VSQU
      RMUL=(UM-VUM)/EM
      TERM1=TERM1/EM
    4 SIGN=-SIGN
      SUM=TERM1
      TERM=TERM1
      EN=0.
    5 EN=EN+1.
      TERM=TERM*VSQ/(EN*(EN+EM))
      SUM=SUM+TERM
      IF (TEST.LE.DABS(RMUL*EN*TERM)) GO TO 5
C     TRUNCATES INNER SERIES IF OUTER TERM*N*INNER TERM < 5.E-7
      SUM1=SUM1+SIGN*RMUL*SUM
      IF (EM.LT..1) GO TO 1
      IF (TEST.LE.DABS(RMUL*SUM)) GO TO 1
C     TRUNCATES OUTER SERIES IF OUTER TERM*INNER SUM < 5.E-7
    6 S(I)=SUM1
      U=UU
      SERIES=S(2)-S(1)
      RETURN
      END
      REAL FUNCTION FCT*8(X)
C     *****************************************************************
C
C     FUNCTION FCT
C
C     PURPOSE
C        TO COMPUTE FCT(X)=EXP(-Z-V**2/(X+Z))/(X+Z)
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
C     DESCRIPTION OF PARAMETERS                                         FCT    8
C        X = THE DOUBLE PRECISION VALUE OF X FOR WHICH FCT IS COMPUTED  FCT    9
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     FCT   10
C        NONE                                                           FCT   11
C     METHOD                                                            FCT   12
C        FORTRAN EVALUATION OF FUNCTION                                 FCT   13
C                                                                       FCT   14
C     ******************************************************************FCT   15
      REAL*8 X,V,Z,P,DEXP                                               FCT   16
      COMMON /C1/ V,Z                                                   FCT   17
      IF (X) 1,2,2                                                      FCT   18
    1 FCT=0.                                                            FCT   19
      GO TO 4                                                           FCT   20
    2 P=Z+V**2/(X+Z)                                                    FCT   21
      IF (P-5.D1) 3,3,1                                                 FCT   22
    3 FCT=DEXP(-P)/(X+Z)                                                FCT   23
    4 RETURN                                                            FCT   24
      END                                                               FCT   25-
      SUBROUTINE DQL12(FCT,Y)                                           DL12 380
C                                                                       DL12  10
C     ..................................................................DL12  20
C                                                                       DL12  30
C        SUBROUTINE DQL12                                               DL12  40
C                                                                       DL12  50
C        PURPOSE                                                        DL12  60
C           TO COMPUTE INTEGRAL(EXP(-X)*FCT(X), SUMMED OVER X           DL12  70
C                               FROM 0 TO INFINITY).                    DL12  80
C                                                                       DL12  90
C        USAGE                                                          DL12 100
C           CALL DQL12 (FCT,Y)                                          DL12 110
C           PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT                DL12 120
C                                                                       DL12 130
C        DESCRIPTION OF PARAMETERS                                      DL12 140
C           FCT    - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION  DL12 150
C                    SUBPROGRAM USED.                                   DL12 160
C           Y      - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.     DL12 170
C                                                                       DL12 180
C        REMARKS                                                        DL12 190
C           NONE                                                        DL12 200
C                                                                       DL12 210
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  DL12 220
C           THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)    DL12 230
C           MUST BE FURNISHED BY THE USER.                              DL12 240
C                                                                       DL12 250
C        METHOD                                                         DL12 260
C           EVALUATION IS DONE BY MEANS OF 12-POINT GAUSSIAN-LAGUERRE   DL12 270
C           QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY,               DL12 280
C           WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23.            DL12 290
C           FOR REFERENCE, SEE                                          DL12 300
C           SHAO/CHEN/FRANK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF    DL12 310
C           CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED     DL12 320
C           GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT       DL12 330
C           TR00.1100 (MARCH 1964), PP.24-25.                           DL12 340
C                                                                       DL12 350
C     ..................................................................DL12 360
C                                                                       DL12 370
C                                                                       DL12 390
C                                                                       DL12 400
      DOUBLE PRECISION X,Y,FCT                                          DL12 410
C                                                                       DL12 420
      X=.3709912104446692 D2                                            DL12 430
      Y=.814807746742624 D-15*FCT(X)                                    DL12 440
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
      X=,2848796725098400 D2
      Y=Y+,3061601635035021 D-11*FCT(X)
      X=,2215109037939701 D2
      Y=Y+,1342391030515004 D-8*FCT(X)
      X=,1711685518746226 D2
      Y=Y+,1668493876540910 D-6*FCT(X)
      X=,1300605499330635 D2
      Y=Y+,8365055856819B0 D-5*FCT(X)
      X=,9621316842456B7 D1
      Y=Y+,2032315926629994 D-3*FCT(X)
      X=,6844525453115177 D1
      Y=Y+,2663973541865316 D-2*FCT(X)
      X=,4599227639418348 D1
      Y=Y+,2010238115463410 D-1*FCT(X)
      X=,2833751337743507 D1
      Y=Y+,9044922221168O9 D-1*FCT(X)
      X=,1512610269776419 D1
      Y=Y+,2440820113198776 D0*FCT(X)
      X=,6117574845151307 D0
      Y=Y+,3777592758731380 D0*FCT(X)
      X=,1157221173580207 D0
      Y=Y+,2647313710554432 D0*FCT(X)
      RETURN
      END
      SUBROUTINE BESK(X,N,BK,IER)
C
C     ..............................................................
C
C        SUBROUTINE BESK
C
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESK(X,N,BK,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED
C           BK -THE RESULTANT K BESSEL FUNCTION
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS ZERO OR NEGATIVE
C              IER=3  X ,GT, 170, MACHINE RANGE EXCEEDED
C              IER=4  BK ,GT, 10**70
C
C        REMARKS
C           N MUST BE GREATER THAN OR EQUAL TO ZERO
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING
C           SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION
C           USING RECURRENCE RELATION,
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE
C           AS DESCRIBED BY A,J,M,HITCHCOCK,'POLYNOMIAL APPROXIMATIONS
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED
C           FUNCTIONS', M,T,A,C,, V,11,1957,PP,86-88, AND G,N, WATSON,
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE
C           UNIVERSITY PRESS, 1958, P, 62
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
C                                                                       BESK 380
C     ..................................................................BESK 390
C                                                                       BESK 400
      DIMENSION T(12)                                                   BESK 420
      BK=.0                                                             BESK 430
      IF(N)10,11,11                                                     BESK 440
   10 IER=1                                                             BESK 450
      RETURN                                                            BESK 460
   11 IF(X)12,12,20                                                     BESK 470
   12 IER=2                                                             BESK 480
      RETURN                                                            BESK 490
   20 IF(X-170.0)22,22,21                                               BESK 500
   21 IER=3                                                             BESK 510
      RETURN                                                            BESK 520
   22 IER=0                                                             BESK 530
      IF(X-1.)36,36,25                                                  BESK 540
   25 A=EXP(-X)                                                         BESK 550
      B=1./X                                                            BESK 560
      C=SQRT(B)                                                         BESK 570
      T(1)=B                                                            BESK 580
      DO 26 L=2,12                                                      BESK 590
   26 T(L)=T(L-1)*B                                                     BESK 600
      IF(N-1)27,29,27                                                   BESK 610
C                                                                       BESK 620
C     COMPUTE K0 USING POLYNOMIAL APPROXIMATION                         BESK 630
C                                                                       BESK 640
   27 G0=A*(1.2533141-.1566642*T(1)+.08811128*T(2)-.09139095*T(3)       BESK 650
     2+.1344596*T(4)-.2299850*T(5)+.3792410*T(6)-.5247277*T(7)          BESK 660
     3+.5575368*T(8)-.4262633*T(9)+.2184518*T(10)-.06680977*T(11)       BESK 670
     4+.009189383*T(12))*C                                              BESK 680
      IF(N)20,28,29                                                     BESK 690
   28 BK=G0                                                             BESK 700
      RETURN                                                            BESK 710
C                                                                       BESK 720
C     COMPUTE K1 USING POLYNOMIAL APPROXIMATION                         BESK 730
C                                                                       BESK 740
   29 G1=A*(1.2533141+.4699927*T(1)-.1468583*T(2)+.1280427*T(3)         BESK 750
     2-.1736432*T(4)+.2847618*T(5)-.4594342*T(6)+.6283381*T(7)          BESK 760
     3-.6632295*T(8)+.5050239*T(9)-.2581304*T(10)+.07880001*T(11)       BESK 770
     4-.01082418*T(12))*C                                               BESK 780
      IF(N-1)20,30,31                                                   BESK 790
   30 BK=G1                                                             BESK 800
      RETURN                                                            BESK 810
C                                                                       BESK 820
C     FROM K0,K1 COMPUTE KN USING RECURRENCE RELATION                   BESK 830
C                                                                       BESK 840
   31 DO 35 J=2,N                                                       BESK 850
      GJ=2.*(FLOAT(J)-1.)*G1/X+G0                                       BESK 860
      IF(GJ-1.0E70)33,33,32                                             BESK 870
   32 IER=4                                                             BESK 880
      GO TO 34                                                          BESK 890
   33 G0=G1                                                             BESK 900
   35 G1=GJ                                                             BESK 910
   34 BK=GJ                                                             BESK 920
      RETURN                                                            BESK 930
   36 B=X/2.                                                            BESK 940
      A=.5772157+ALOG(B)                                                BESK 950
      C=B*B                                                             BESK 960
      IF(N-1)37,43,37                                                   BESK 970
C                                                                       BESK 980
C     COMPUTE K0 USING SERIES EXPANSION                                 BESK 990
```

TABLE 4.3—*Listing of program for radial flow in a leaky artesian aquifer*—Continued

```
C                                                                       BESK1000
   37 G0=-A                                                             BESK1010
      X2J=1.                                                            BESK1020
      FACT=1.                                                           BESK1030
      HJ=.0                                                             BESK1040
      DO 40 J=1,6                                                       BESK1050
      RJ=1./FLOAT(J)                                                    BESK1060
      IF(X2J.LT.1.E-40) X2J=0.                                          BESK1061
C     PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW   BESK1062
C     PROBLEM ON WATFOR COMPILER                                        BESK1063
      X2J=X2J*C                                                         BESK1070
      FACT=FACT*RJ*RJ                                                   BESK1080
      HJ=HJ+RJ                                                          BESK1090
   40 G0=G0+X2J*FACT*(HJ-A)                                             BESK1100
      IF(N)43,42,43                                                     BESK1110
   42 BK=G0                                                             BESK1120
      RETURN                                                            BESK1130
C                                                                       BESK1140
C     COMPUTE K1 USING SERIES EXPANSION                                 BESK1150
C                                                                       BESK1160
   43 X2J=B                                                             BESK1170
      FACT=1.                                                           BESK1180
      HJ=1.                                                             BESK1190
      G1=1./X+X2J*(.5+A-HJ)                                             BESK1200
      DO 50 J=2,8                                                       BESK1210
      X2J=X2J*C                                                         BESK1220
      RJ=1./FLOAT(J)                                                    BESK1230
      FACT=FACT*RJ*RJ                                                   BESK1240
      HJ=HJ+RJ                                                          BESK1250
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))                               BESK1260
      IF(N-1)31,52,31                                                   BESK1270
   52 BK=G1                                                             BESK1280
      RETURN                                                            BESK1290
      END                                                               BESK130-
```

TABLE 5.2—*Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds*

```
C     ******************************************************************LST    1
C                                                                       LST    2
C     PURPOSE                                                           LST    3
C       TO COMPUTE TYPE CURVE FUNCTION VALUES FOR H(U,BETA) --          LST    4
C       HANTUSH,M.S.,1960, MODIFICATION OF THE THEORY OF LEAKY          LST    5
C       AQUIFERS: JOUR. GEOPHYS. RES., V. 65, NO. 11,  P. 3713-3725.   LST    6
C       THE COMPUTATIONAL ALGORITHM WAS DEVISED AND PROGRAMMED BY       LST    7
C       S.S.PAPADOPULOS.                                                LST    8
C     INPUT DATA                                                        LST    9
C       1 CARD - FORMAT(2E10.5)                                         LST   10
C          USMALL - SMALLEST(BEGINNING) VALUE OF 1/U.                   LST   11
C          ULARGE - LARGEST(ENDING) VALUE OF 1/U.                       LST   12
C       2 CARDS - FORMAT(8E10.5)                                        LST   13
C          BDAT - 12 VALUES OF BETA (ZERO OR BLANK VALUES ARE           LST   14
C          PERMISSIBLE IF LESS THAN 12 DESIRED, WILL TERMINATE         LST   15
C          AT FIRST ZERO OR BLANK VALUE).                               LST   16
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     LST   17
C       H,DQG32,MUH,W - MUST BE INCLUDED IN DECK.                       LST   18
C       DSQRT,DEXP,DERFC,DLOG - MUST BE IN COMPUTER LIBRARY.            LST   19
C                                                                       LST   20
C     ******************************************************************LST   21
      REAL*8 U,BETA,H                                                   LST   22
      DIMENSION ARRAY(73,12), Y(73), BDAT(12), YNUM(6)                  LST   23
      DATA YNUM/1.,1.5,2.,3.,5.,7./                                     LST   24
      IRD=5                                                             LST   25
      IPT=6                                                             LST   26
      READ (IRD,6) USMALL,ULARGE                                        LST   27
      READ (IRD,6) BDAT                                                 LST   28
      IBEGIN=ALOG10(USMALL)                                             LST   29
      IEND=ALOG10(ULARGE)+.99999                                        LST   30
      ILIMIT=(IEND-IBEGIN)*6+1                                          LST   31
      IF (ILIMIT.GT.73) ILIMIT=73                                       LST   32
      DO 1 I=1,12                                                       LST   33
      IF (BDAT(I).EQ.0.) GO TO 2                                        LST   34
    1 CONTINUE                                                          LST   35
      NB=12                                                             LST   36
```

TABLE 5.2—*Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds*—Continued

```
      GO TO 3
    2 NB=I-1
      II=0
    3 DO 4 I=1,ILIMIT
      IEXP=IBEGIN+(I-1)/6
      II=II+1
      IF (II,GT,6) II=1
      Y(I)=YNUM(II)*10,**IEXP
      U=1,/Y(I)
      DO 4 J=1,NB
      BETA=BDAT(J)
    4 ARRAY(I,J)=H(U,BETA)
      WRITE (IPT,7) (BDAT(I),I=1,NB)
      DO 5 I=1,ILIMIT
    5 WRITE (IPT,8) Y(I),(ARRAY(I,J),J=1,NB)
      STOP
C
    6 FORMAT (8E10,5)
    7 FORMAT ('1','H(U,BETA)'/'0',10X,'| BETA'/' ',6X,'1/U |',12E10,2)
    8 FORMAT (' ',E10,3,12F10,4)
      END
      DOUBLE PRECISION FUNCTION H(U,B)
C     ******************************************************************
C
C     FUNCTION H
C     PURPOSE
C        TO COMPUTE THE INTEGRAL OF
C        EXP(-Y)*ERFC(B*SQRT(U)/SQRT(Y*(Y-U)))/Y SUMMED OVER Y
C        FROM U TO INFINITY (FUNCTION H(U,BETA) OF HANTUSH),
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U = R**2*S/(4*T*TIME), (RADIAL DISTANCE SQUARED * STORAGE
C        COEFFICIENT / (4 * TRANSMISSIVITY * TIME), U MUST BE > 1,D-60,
C        B = (R/4)*(SQRT(K'*S'/(B'*T*S)+K''*S''/(B''*T*S)),
C             K',S',B' = HYD, COND,, STORAGE COEFF,, THICKNESS OF
C                  UPPER CONFINING BED,
C             K'',S'',B'' = HYD, COND,, STORAGE COEFF,, THICKNESS OF
C                  LOWER CONFINING BED,
C     METHOD
C        I, FOR U < 1,D-60, NO COMPUTATION IS MADE,
C        II, FOR B=0, H(U,0)=W(U) (THEIS WELL FUNCTION),
C        III, H(U,B)=0 IF
C             1, U > 10,
C             2, B > 1 AND B**2*U > 300,
C        IV, ERFC(ARG)=0 FOR ARG > 40 AND H(U,B) = H(UB,B)
C             FOR U < Y < UB WHERE UB IS THE U CORRESPONDING TO ARG = 40
C             SINCE H(UB,B) < W(UB) THEN FOR UB > 10, H(U,B) = 0,
C             ERFC(ARG) = 1 FOR ARG < 2,E-10 AND H(UUB,B) = W(UUB)
C             WHERE UUB IS THE U CORRESPONDING TO ARG = 2,E-10,
C             IF UUB > 10, H(U,B) = INTEGRAL FROM UB TO 10,
C             IF UUB < 10, H(U,B) = INTEGRAL FROM UB TO UUB + W(UUB)
C
C     ******************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON UUU,BBB
      EXTERNAL HUB
      UUU=U
      BBB=B
      IF (U,GT,1,D-60) GO TO 1
      WRITE (6,7)
      STOP
    1 IF (B,EQ,0,0) GO TO 5
      IF (U,GT,10,0) GO TO 6
      BU=B*B*U
      IF (B,GT,1,0,AND,BU,GE,300,0) GO TO 6
      H1=0,0
      UP=10,0
      UB=0,5*U*(1,0+DSQRT(1,0+0,025*B*B/U))
      IF (UB,GT,UP) GO TO 6
      UUB=0,5*U*(1,0+DSQRT(1,0+1,D20*B*B/U))
      IF (UUB,GT,UP) GO TO 2
      H1=W(UUB)
      UP=UUB
    2 H2=0,0
      XL=UB
    3 XU=10,*XL
      IF (XU,GE,UP) XU=UP
      CALL DQG32(XL,XU,HUB,AREA)
      H2=H2+AREA
      XL=XU
      IF (XL,EQ,UP) GO TO 4
      GO TO 3
    4 H=H1+H2
      RETURN
    5 H=W(U)
      RETURN
```

TABLE 5.2—*Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds—Continued*

```
      6 H=0,0                                                              H   66
        RETURN                                                             H   67
C                                                                          H   68
      7 FORMAT ('0','U TOO SMALL FOR COMPUTATION')                         H   69
        END                                                                H   70-

        SUBROUTINE DQG32(XL,XU,FCT,Y)                                      DQG  1
C                                                                          DQG  2
C       ...................................................................DQG  3
C                                                                          DQG  4
C          SUBROUTINE DQG32                                                DQG  5
C                                                                          DQG  6
C          PURPOSE                                                         DQG  7
C             TO COMPUTE INTEGRAL(FCT(X), SUMMED OVER X FROM XL TO XU)     DQG  8
C                                                                          DQG  9
C          USAGE                                                           DQG 10
C             CALL DQG32 (XL,XU,FCT,Y)                                     DQG 11
C             PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT                 DQG 12
C                                                                          DQG 13
C          DESCRIPTION OF PARAMETERS                                       DQG 14
C             XL     - DOUBLE PRECISION LOWER BOUND OF THE INTERVAL.       DQG 15
C             XU     - DOUBLE PRECISION UPPER BOUND OF THE INTERVAL.       DQG 16
C             FCT    - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION   DQG 17
C                      SUBPROGRAM USED.                                    DQG 18
C             Y      - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.      DQG 19
C                                                                          DQG 20
C          REMARKS                                                         DQG 21
C             NONE                                                         DQG 22
C                                                                          DQG 23
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                   DQG 24
C             THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)     DQG 25
C             MUST BE FURNISHED BY THE USER,                               DQG 26
C                                                                          DQG 27
C          METHOD                                                          DQG 28
C             EVALUATION IS DONE BY MEANS OF 32-POINT GAUSS QUADRATURE     DQG 29
C             FORMULA, WHICH INTEGRATES POLYNOMIALS UP TO DEGREE 63        DQG 30
C             EXACTLY, FOR REFERENCE, SEE                                  DQG 31
C             V,I,KRYLOV, APPROXIMATE CALCULATION OF INTEGRALS,            DQG 32
C             MACMILLAN, NEW YORK/LONDON, 1962, PP,100-111 AND 337-340,    DQG 33
C                                                                          DQG 34
C       ...................................................................DQG 35
C                                                                          DQG 36
        DOUBLE PRECISION XL,XU,Y,A,B,C,FCT                                 DQG 37
        A=,5D0*(XU+XL)                                                     DQG 38
        B=XU-XL                                                            DQG 39
        C=,49863193092474080D0*B                                           DQG 40
        Y=,35093050047350480D-2*(FCT(A+C)+FCT(A-C))                        DQG 41
        C=,49280575577263420D0*B                                           DQG 42
        Y=Y+,81371973654528340D-2*(FCT(A+C)+FCT(A-C))                      DQG 43
        C=,48238112779375320D0*B                                           DQG 44
        Y=Y+,12696032654631030D-1*(FCT(A+C)+FCT(A-C))                      DQG 45
        C=,46745303796886980D0*B                                           DQG 46
        Y=Y+,17136931456510720D-1*(FCT(A+C)+FCT(A-C))                      DQG 47
        C=,44816057788302610D0*B                                           DQG 48
        Y=Y+,21417949011113340D-1*(FCT(A+C)+FCT(A-C))                      DQG 49
        C=,42468380686628500D0*B                                           DQG 50
        Y=Y+,25499029631188090D-1*(FCT(A+C)+FCT(A-C))                      DQG 51
        C=,39724189798397120D0*B                                           DQG 52
        Y=Y+,29342046739267770D-1*(FCT(A+C)+FCT(A-C))                      DQG 53
        C=,36609105937014480D0*B                                           DQG 54
        Y=Y+,32911111388180920D-1*(FCT(A+C)+FCT(A-C))                      DQG 55
        C=,33152213346510760D0*B                                           DQG 56
        Y=Y+,36172897054424250D-1*(FCT(A+C)+FCT(A-C))                      DQG 57
        C=,29385787862038120D0*B                                           DQG 58
        Y=Y+,39096947893535150D-1*(FCT(A+C)+FCT(A-C))                      DQG 59
        C=,25344995446611470D0*B                                           DQG 60
        Y=Y+,41655962113473380D-1*(FCT(A+C)+FCT(A-C))                      DQG 61
        C=,21067563806531770D0*B                                           DQG 62
        Y=Y+,43826046502201910D-1*(FCT(A+C)+FCT(A-C))                      DQG 63
        C=,16593430114106380D0*B                                           DQG 64
        Y=Y+,45586939347881940D-1*(FCT(A+C)+FCT(A-C))                      DQG 65
        C=,11964368112606850D0*B                                           DQG 66
        Y=Y+,46922199540402280D-1*(FCT(A+C)+FCT(A-C))                      DQG 67
        C=,72235980791398200-1*B                                           DQG 68
        Y=Y+,47819360039637430D-1*(FCT(A+C)+FCT(A-C))                      DQG 69
        C=,24153832843869160D-1*B                                          DQG 70
        Y=B*(Y+,48270044257363900D-1*(FCT(A+C)+FCT(A-C)))                  DQG 71
        RETURN                                                             DQG 72
        END                                                                DQG 73-
        DOUBLE PRECISION FUNCTION HUB(X)                                   HUB  1
C       *******************************************************************HUB  2
C                                                                          HUB  3
C       FUNCTION HUB                                                       HUB  4
C       PURPOSE                                                            HUB  5
C          TO COMPUTE VALUES OF THE INTEGRAND OF H(U,B)                    HUB  6
C       DESCRIPTION OF PARAMETER                                           HUB  7
C          X - DOUBLE PRECISION, POINT AT WHICH INTEGRAND IS EVALUATED,    HUB  8
```

TABLE 5.2—*Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds*—Continued

```
C      METHOD                                                            HUB   9
C         FORTRAN EVALUATION OF FUNCTION.                                HUB  10
C                                                                        HUB  11
C      ******************************************************************HUB  12
       IMPLICIT REAL*8(A-H,O-Z)                                          HUB  13
       COMMON UUU,BBB                                                    HUB  14
       ARG=DSQRT((BBB*BBB*UUU)/(X*X-X*UUU))                              HUB  15
       HUB=DEXP(-X)*DERFC(ARG)/X                                         HUB  16
       RETURN                                                            HUB  17
       END                                                               HUB  18-

       DOUBLE PRECISION FUNCTION W(U)                                     WU   1
C      *****************************************************************  WU   2
C                                                                         WU   3
C      FUNCTION W                                                         WU   4
C      PURPOSE                                                            WU   5
C         TO EVALUATE THE WELL FUNCTION OF THEIS.                         WU   6
C      DESCRIPTION OF PARAMETER                                           WU   7
C         U - DOUBLE PRECISION, ARGUMENT FOR WELL FUNCTION.               WU   8
C                                                                         WU   9
C      *****************************************************************  WU  10
       IMPLICIT REAL*8 (A-H,O-Z)                                          WU  11
       IF (U.LE.0.0) GO TO 2                                              WU  12
       IF (U.GT.100.) GO TO 3                                             WU  13
       IF (U.GE.1.0) GO TO 1                                              WU  14
       W=-.57721566+U*(.99999193+U*(-.24991055+U*(.05519968+U*(-.00976004 WU  15
      1+.00107857*U))))-DLOG(U)                                           WU  16
       GO TO 4                                                            WU  17
     1 ENUM=DEXP(-U)*(.2677737343+U*(8.6347608925+U*(18.0590169730+U*(8.5 WU  18
      1733287401+U))))                                                    WU  19
       DEN=U*(3.9584969228+U*(21.0996530827+U*(25.6329561486+U*(9.5733223 WU  20
      1454+U))))                                                          WU  21
       W=ENUM/DEN                                                         WU  22
       GO TO 4                                                            WU  23
     2 WRITE (6,5) U                                                      WU  24
       STOP                                                               WU  25
     3 W=0.0                                                              WU  26
     4 RETURN                                                             WU  27
C                                                                         WU  28
     5 FORMAT ('0',5X,'W(U) NOT DEFINED FOR U=',1PD15.8)                  WU  29
       END                                                                WU  30-
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*

```
C      ******************************************************************PPL   1
C                                                                        PPL   2
C      PURPOSE                                                           PPL   3
C         TO COMPUTE TYPE CURVE FUNCTION VALUES FOR PARTIAL PENETRATION  PPL   4
C         IN A LEAKY AQUIFER USING EQ. 73 OF HANTUSH,M.S., 1964,         PPL   5
C         HYDRAULICS OF WELLS IN CHOW, VEN TE, ADVANCES IN HYDROSCIENCE, PPL   6
C         VOL. 1: ACADEMIC PRESS, NEW YORK, P. 281-442.                  PPL   7
C      INPUT DATA                                                        PPL   8
C         1 CARD  -  FORMAT (3F5.1,I5,2E10.4)                            PPL   9
C              B - AQUIFER THICKNESS                                     PPL  10
C              E - DEPTH, BELOW TOP OF AQUIFER, TO BOTTOM OF PUMPING     PPL  11
C              WELL SCREEN                                               PPL  12
C              D - DEPTH, BELOW TOP OF AQUIFER, TO TOP OF PUMPING WELL   PPL  13
C              SCREEN                                                    PPL  14
C              NUM - NUMBER OF OBSERVATION WELLS OR PIEZOMETERS TIMES    PPL  15
C              NUMBER OF VALUES OF KZ/KR.                                PPL  16
C              SMALL - SMALLEST VALUE OF 1/U FOR WHICH COMPUTATION IS    PPL  17
C              DESIRED                                                   PPL  18
C              LARGE - LARGEST  VALUE OF 1/U FOR WHICH COMPUTATION IS    PPL  19
C              DESIRED                                                   PPL  20
C         2 CARDS - FORMAT(8E10.5)                                       PPL  21
C              BOAT - 12 VALUES OF R/BR, NON ZERO VALUES SHOULD BE       PPL  22
C              FIRST, WILL TERMINATE AT FIRST ZERO (OR BLANK) VALUE.     PPL  23
C         NUM CARDS (ONE FOR EACH OBS. WELL OR PIEZOMETER AND FOR EACH   PPL  24
C              VALUE OF R*SQRT(KZ/KR) - FORMAT (3F5.1)                   PPL  25
C              R - RADIAL DISTANCE FROM PUMPED WELL TIMES SQRT(KZ/KR),   PPL  26
C              LPRIME - DEPTH, BELOW TOP OF AQUIFER, TO BOTTOM OF OBS.   PPL  27
C              WELL SCREEN (ZERO FOR PIEZOMETER)                         PPL  28
C              DPRIME - DEPTH, BELOW TOP OF AQUIFER, TO TOP OF OBS. WELL PPL  29
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
C            SCREEN (TOTAL DEPTH FOR PIEZOMETER)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT,L,FL
C
C     ******************************************************************
      REAL*8 U,V
      REAL*4 L,LB,LPB,LPRIME,LARGE
      DIMENSION ARRAY(55,12), ARG(6), BDAT(12), Y(55)
      DATA ARG/1.,1.5,2.,3.,5.,7./
      DATA ARRAY/660*0./,Y/55*0./
      IRD=5
      IPT=6
      READ (IRD,9) B,E,D,NUM,SMALL,LARGE
      READ (IRD,14) BDAT
      DO 1 I=1,12
      IF (BDAT(I).EQ.0.) GO TO 2
    1 CONTINUE
      NB=12
      GO TO 3
    2 NB=I-1
    3 LB=E/B
      DB=D/B
      IBEGIN=ALOG10(SMALL)
      IEND=ALOG10(LARGE)+.1
      JLIMIT=IEND-IBEGIN
      IF (JLIMIT.GT.9) JLIMIT=9
      ILIMIT=6*JLIMIT+1
      DO 8 K=1,NUM
      READ (IRD,9) R,LPRIME,DPRIME
      RB=R/B
      LPB=LPRIME/B
      DPB=DPRIME/B
      DO 4 I=1,ILIMIT
      INDEX=(I-1)/6
      IEXP=IBEGIN+INDEX
      II=I-INDEX*6
      Y(I)=ARG(II)*10.**IEXP
      U=1./Y(I)
      DO 4 J=1,NB
      BETA=BDAT(J)
      V=BETA/2.
    4 ARRAY(I,J)=L(U,V)+FL(U,RB,BETA,LB,DB,LPB,DPB)
      IF (LPB-0.) 5,5,6
    5 WRITE (IPT,10) DPB,RB,LB,DB
      GO TO 7
    6 WRITE (IPT,11) LPB,DPB,RB,LB,DB
    7 WRITE (IPT,12) (BDAT(I),I=1,NB)
      DO 8 I=1,ILIMIT
      WRITE (IPT,13) Y(I),(ARRAY(I,J),J=1,NB)
    8 CONTINUE
      STOP
C
C
    9 FORMAT (3F5.1,I5,2E10.4)
   10 FORMAT ('1','W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,Z/B), Z/B=',F5.2,', SQ
     1RT(KZ/KR)*R/B=',F5.2,', L/B=',F5.2,', D/B=',F5.2)
   11 FORMAT ('1','W(U,R/BR)+F(U,R/B,R/BR,L/B,D/B,L''/B,D''/B), L''/B=',
     1F5.2,', D''/B=',F5.2,', SQRT(KZ/KR)*R/B=',F5.2,', L/B=',F5.2,', D/
     2B=',F5.2)
   12 FORMAT ('0',9X,'| R/BR'/' ',5X,'1/U |',12E10.2)
   13 FORMAT (' ',E10.3,12F10.4)
   14 FORMAT (8E10.5)
      END
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
      REAL FUNCTION FL*4(U,RB,BETA,LB,DB,LPB,DPB)
C     ******************************************************************
C
C     FUNCTION FL
C
C     PURPOSE
C        TO COMPUTE DEPARTURES FROM HANTUSH-JACOB LEAKY AQUIFER CURVE
C        CAUSED BY PARTIAL PENETRATION OF PUMPED WELL.
C     USAGE
C        FL(U,RB,BETA,LB,DB,LPB,DPB)
C     DESCRIPTION OF PARAMETERS
C        ALL REAL, U DOUBLE PRECISION
C        U = R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        RB = R/B ( RADIAL DISTANCE / AQUIFER THICKNESS )
C        BETA = R*SQRT(K'/B'T) = (RADIAL DISTANCE * SQUARE ROOT
C               (HYD. COND. OF CONFINING BED/THICKNESS OF CONFINING
C               BED * TRANSMISSIVITY OF AQUIFER))
C        LB = L/B ( FRACTION OF AQUIFER PENETRATED BY PUMPED WELL)
C        DB = D/B ( FRACTION OF AQUIFER ABOVE PUMPED WELL SCREEN)
C        LPB = L'/B (FRACTION OF AQUIFER PENETRATED BY OBS. WELL, ZERO
C              FOR PIEZOMETER)
C        DPB = D'/B (FRACTION OF AQUIFER ABOVE OBS. WELL SCREEN, TOTAL
C              DEPTH FOR PIEZOMETER)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        DQL12,SERIES,BESK,FCT,L
C     METHOD
C        SUMS THE SERIES THROUGH N*PI*R/B EQ 20
C
C     ******************************************************************
      REAL*8 U,V,DSQRT
      REAL*4 L,N,LB,LPB
      SUM=0.
      N=0.
      BETSQ=BETA*BETA
      PIRBSQ=9.869604*RB*RB
      PILB=3.141593*LB
      PIDB=3.141593*DB
      IF (LPB-0.) 1,1,4
C     CHECKS FOR WELL OR PIEZOMETER
    1 PIZB=3.141593*DPB
    2 N=N+1.
      V=SQRT(BETSQ+N*N*PIRBSQ)/2.
      IF (V.GT.10.) GO TO 3
C     TRUNCATES SERIES WHEN V>10
      X=L(U,V)/N
      SUM=SUM+(SIN(N*PILB)-SIN(N*PIDB))*COS(N*PIZB)*X
      GO TO 2
    3 FL=.6366198*SUM/(LB-DB)
      GO TO 7
    4 PILPB=3.141593*LPB
      PIDPB=3.141593*DPB
    5 N=N+1
      V=SQRT(BETSQ+N*N*PIRBSQ)/2.
      IF (V.GT.10.) GO TO 6
C     TRUNCATES SERIES WHEN V>10
      X=L(U,V)/N
      SUM=SUM+(SIN(N*PILB)-SIN(N*PIDB))*(SIN(N*PILPB)-SIN(N*PIDPB))*X/N
      GO TO 5
    6 FL=.2026424*SUM/((LB-DB)*(LPB-DPB))
    7 RETURN
      END
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
      REAL FUNCTION L*4(U,V)
C     *****************************************************************
C
C     FUNCTION L
C
C     PURPOSE
C        TO COMPUTE THE INTEGRAL( EXP(-Y-V**2/Y)/Y) SUMMED OVER Y FROM
C        U TO INFINITY(WELL FUNCTION FOR LEAKY AQUIFERS).
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U = R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V = R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED\
C        DQL12,SERIES,BESK,FCT
C     METHOD
C        IN THE FOLLOWING F=EXP(-Y-V**2/Y)/Y
C      (1) U>=1, USES A GAUSSIAN-LAGUERRE QUADRATURE FORMULA TO
C        EVALUATE INTEGRAL(F) FROM U TO INF.
C      (2) V**2<U<1, USES THE G-L QUADRATURE TO EVALUATE INTEGRAL(F)
C        FROM ONE TO INF AND A SERIES EXPANSION TO EVALUATE INTEGRAL(F)
C        FROM U TO ONE.
C      (3) U<1, U<=V**2, USES THE REPRESENTATION INTEGRAL(F) FROM U
C        TO INF. = 2*K0(2*V)-INTEGRAL(F) FROM V**2/U TO INF.
C        EVALUATES THE ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND
C        KIND WITH IBM SUBROUTINE, EVALUATES INTEGRAL BY G-L QUAD.
C
C     *****************************************************************
      EXTERNAL FCT
      REAL*8 U,V,Z,F,VV,SERIES
      COMMON /C1/ VV,Z
      VV=V
      IF (U-1.) 1,2,2
C     CHECKS IF U<1
    1 Z=V*V/U
      IF (Z-1.) 3,4,4
C     CHECKS IF V**2/U < 1
    2 Z=U
      CALL DQL12(FCT,F)
      L=F
C     INTEGRAL U TO INF. EVALUATED BY GAUSS-LAGUERRE QUADRATURE
      GO TO 5
    3 Z=1.
      CALL DQL12(FCT,F)
      L=F+SERIES(U,V)
C     INTEGRAL 1 TO INF. BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP.
      GO TO 5
    4 TWOV=2.*V
      CALL BESK(TWOV,0,BK,IER)
      CALL DQL12(FCT,F)
      L=2.*BK-F
C     2K0(2V)-INTEGRAL V**2/U TO INF.
    5 RETURN
      END
      REAL FUNCTION SERIES*8(U,V)
C     *******************************************************************
C
C     FUNCTION SERIES
C
C     PURPOSE
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
C        TO EVALUATE S(1)-S(U), WHERE S IS A SERIES EXPANSION OF
C        INTEGRAL(EXP(-Y-V**2/Y)DY/Y) GIVEN BY: S= SUM, M=0 TO INFINITY,
C        (F(M)*SUM, N=0 TO INF.,(V**(2*N)/((N!)*(M+N)!)) WHERE F(M)=
C        -LOG(U) IF M=0 AND = ((-1)**M/M)*(U**M-(V**2/U)**M) IF M>0.
C     DESCRIPTION OF PARAMETERS
C        BOTH DOUBLE PRECISION
C        U = R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
C            COEFFICIENT / 4*TRANSMISSIVITY * TIME
C        V = R/2*SQRT(K'/(T*B'))--ONE-HALF RADIAL DISTANCE*SQUARE ROOT
C            (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS
C            OF CONFINING BED)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE
C     METHOD
C        SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7/N AND FOR OUTER SERIES WHEN A TERM
C        BECOMES LESS THAN 5.E-7
C
C     *****************************************************************
      REAL*8 DLOG,DABS,S(2),VUM,UU
      REAL*8 TEST,U,UM,EM,EN,SUM1,SUM,SIGN,V,VSQ,VSQU,RMUL,TERM,TERM1
      TEST=5.D-07
      VSQ=V*V
      UU=U
      DO 6 I=1,2
C     EVALUATES SERIES FOR LOWER LIMIT = U AND UPPER LIMIT = 1
      IF (I.EQ.2) U=1.
      UM=1.
      EM=-1.
      SUM1=0.
      SIGN=-1.
      VUM=1.
      VSQU=VSQ/U
    1 EM=EM+1.
      IF (EM-.1) 2,3,3
C     CHECKS FOR M=0
    2 RMUL=DLOG(U)
      TERM1=1.
      GO TO 4
    3 UM=UM*U
      IF (VUM.LT.1.D-30) VUM=0.
      VUM=VUM*VSQU
      RMUL=(UM-VUM)/EM
      TERM1=TERM1/EM
    4 SIGN=-SIGN
      SUM=TERM1
      TERM=TERM1
      EN=0.
    5 EN=EN+1.
      TERM=TERM*VSQ/(EN*(EN+EM))
      SUM=SUM+TERM
      IF (TEST.LE.DABS(RMUL*EN*TERM)) GO TO 5
C     TRUNCATES INNER SERIES IF OUTER TERM*N*INNER TERM < 5.E-7
      SUM1=SUM1+SIGN*RMUL*SUM
      IF (EM.LT..1) GO TO 1
      IF (TEST.LE.DABS(RMUL*SUM)) GO TO 1
C     TRUNCATES OUTER SERIES IF OUTER TERM*INNER SUM < 5.E-7
    6 S(I)=SUM1
      U=UU
      SERIES=S(2)-S(1)
      RETURN
      END
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
      REAL FUNCTION FCT*8(X)                                            FCT    1
C     ******************************************************************FCT    2
C                                                                       FCT    3
C     FUNCTION FCT                                                      FCT    4
C                                                                       FCT    5
C     PURPOSE                                                           FCT    6
C        TO COMPUTE FCT(X)=EXP(-Z-V**2/(X+Z))/(X+Z)                     FCT    7
C     DESCRIPTION OF PARAMETERS                                         FCT    8
C        X - THE DOUBLE PRECISION VALUE OF X FOR WHICH FCT IS COMPUTED  FCT    9
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     FCT   10
C        NONE                                                           FCT   11
C     METHOD                                                            FCT   12
C        FORTRAN EVALUATION OF FUNCTION                                 FCT   13
C                                                                       FCT   14
C     ******************************************************************FCT   15
      REAL*8 X,V,Z,P,DEXP                                               FCT   16
      COMMON /C1/ V,Z                                                   FCT   17
      IF (X) 1,2,2                                                      FCT   18
    1 FCT=0.                                                            FCT   19
      GO TO 4                                                           FCT   20
    2 P=Z+V**2/(X+Z)                                                    FCT   21
      IF (P-5.01) 3,3,1                                                 FCT   22
    3 FCT=DEXP(-P)/(X+Z)                                                FCT   23
    4 RETURN                                                            FCT   24
      END                                                               FCT   25-
      SUBROUTINE DQL12(FCT,Y)                                           DL12 380
C                                                                       DL12  10
C     ..................................................................DL12  20
C                                                                       DL12  30
C        SUBROUTINE DQL12                                               DL12  40
C                                                                       DL12  50
C        PURPOSE                                                        DL12  60
C           TO COMPUTE INTEGRAL(EXP(-X)*FCT(X), SUMMED OVER X           DL12  70
C                                FROM 0 TO INFINITY).                   DL12  80
C                                                                       DL12  90
C        USAGE                                                          DL12 100
C           CALL DQL12 (FCT,Y)                                          DL12 110
C           PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT                DL12 120
C                                                                       DL12 130
C        DESCRIPTION OF PARAMETERS                                      DL12 140
C           FCT    - THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION  DL12 150
C                    SUBPROGRAM USED.                                   DL12 160
C           Y      - THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.     DL12 170
C                                                                       DL12 180
C        REMARKS                                                        DL12 190
C           NONE                                                        DL12 200
C                                                                       DL12 210
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  DL12 220
C           THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)    DL12 230
C           MUST BE FURNISHED BY THE USER.                              DL12 240
C                                                                       DL12 250
C        METHOD                                                         DL12 260
C           EVALUATION IS DONE BY MEANS OF 12-POINT GAUSSIAN-LAGUERRE   DL12 270
C           QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY,               DL12 280
C           WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23.            DL12 290
C           FOR REFERENCE, SEE                                          DL12 300
C           SHAO/CHEN/FRANK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF    DL12 310
C           CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED     DL12 320
C           GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT       DL12 330
C           TR00.1100 (MARCH 1964), PP.24-25.                           DL12 340
C                                                                       DL12  50
C     ..................................................................DL12 360
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
C                                                                       DL12 370
C                                                                       DL12 390
C                                                                       DL12 400
      DOUBLE PRECISION X,Y,FCT                                          DL12 410
C                                                                       DL12 420
      X=.3709912104446692 D2                                            DL12 430
      Y=.814807746742624 D-15*FCT(X)                                    DL12 440
      X=.2848796725098400 D2                                            DL12 450
      Y=Y+.3061601635035021 D-11*FCT(X)                                 DL12 460
      X=.2215109037939701 D2                                            DL12 470
      Y=Y+.1342391030515004 D-8*FCT(X)                                  DL12 480
      X=.1711685518746226 D2                                            DL12 490
      Y=Y+.1668493876540910 D-6*FCT(X)                                  DL12 500
      X=.1300605499330635 D2                                            DL12 510
      Y=Y+.836505585681980 D-5*FCT(X)                                   DL12 520
      X=.962131684245687 D1                                             DL12 530
      Y=Y+.2032315926629994 D-3*FCT(X)                                  DL12 540
      X=.6844525453115177 D1                                            DL12 550
      Y=Y+.2663973541865316 D-2*FCT(X)                                  DL12 560
      X=.4599227639418348 D1                                            DL12 570
      Y=Y+.2010238115463410 D-1*FCT(X)                                  DL12 580
      X=.2833751337743507 D1                                            DL12 590
      Y=Y+.904492222116809 D-1*FCT(X)                                   DL12 600
      X=.1512610269776419 D1                                            DL12 610
      Y=Y+.2440820113198776 D0*FCT(X)                                   DL12 620
      X=.6117574845151307 D0                                            DL12 630
      Y=Y+.3777592758731380 D0*FCT(X)                                   DL12 640
      X=.1157221173580207 D0                                            DL12 650
      Y=Y+.2647313710554432 D0*FCT(X)                                   DL12 660
      RETURN                                                            DL12 670
      END                                                               DL12 68-
      SUBROUTINE BESK(X,N,BK,IER)                                       BESK 410
C                                                                       BESK  10
C     ..................................................................BESK  20
C                                                                       BESK  30
C        SUBROUTINE BESK                                                BESK  40
C                                                                       BESK  50
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDERBESK  60
C                                                                       BESK  70
C        USAGE                                                          BESK  80
C           CALL BESK(X,N,BK,IER)                                       BESK  90
C                                                                       BESK 100
C        DESCRIPTION OF PARAMETERS                                      BESK 110
C           X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED           BESK 120
C           N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED              BESK 130
C           BK -THE RESULTANT K BESSEL FUNCTION                         BESK 140
C           IER-RESULTANT ERROR CODE WHERE                              BESK 150
C              IER=0  NO ERROR                                          BESK 160
C              IER=1  N IS NEGATIVE                                     BESK 170
C              IER=2  X IS ZERO OR NEGATIVE                             BESK 180
C              IER=3  X .GT. 170, MACHINE RANGE EXCEEDED                BESK 190
C              IER=4  BK .GT. 10**70                                    BESK 200
C                                                                       BESK 210
C        REMARKS                                                        BESK 220
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                     BESK 230
C                                                                       BESK 240
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  BESK 250
C           NONE                                                        BESK 260
C                                                                       BESK 270
C        METHOD                                                         BESK 280
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING  BESK 290
C           SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION BESK 300
C           USING RECURRENCE RELATION.                                  BESK 310
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE  BESK 320
C           AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS  BESK 330
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED    BESK 340
C           FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,  BESK 350
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE   BESK 360
C           UNIVERSITY PRESS, 1958, P. 62                               BESK 370
C                                                                       BESK 380
C     ..................................................................BESK 390
C                                                                       BESK 400
      DIMENSION T(12)                                                   BESK 420
      BK=.0                                                             BESK 430
      IF(N)10,11,11                                                     BESK 440
   10 IER=1                                                             BESK 450
      RETURN                                                            BESK 460
   11 IF(X)12,12,20                                                     BESK 470
   12 IER=2                                                             BESK 480
      RETURN                                                            BESK 490
   20 IF(X-170.0)22,22,21                                               BESK 500
   21 IER=3                                                             BESK 510
      RETURN                                                            BESK 520
   22 IER=0                                                             BESK 530
      IF(X-1.)36,36,25                                                  BESK 540
   25 A=EXP(-X)                                                         BESK 550
      B=1./X                                                            BESK 560
      C=SQRT(B)                                                         BESK 570
      T(1)=B                                                            BESK 580
      DO 26 L=2,12                                                      BESK 590
   26 T(L)=T(L-1)*B                                                     BESK 600
      IF(N-1)27,29,27                                                   BESK 610
C                                                                       BESK 620
C     COMPUTE K0 USING POLYNOMIAL APPROXIMATION                         BESK 630
C                                                                       BESK 640
   27 G0=A*(1.2533141-.1566642*T(1)+.08811128*T(2)-.09139095*T(3)       BESK 650
     2+.1344596*T(4)-.2299850*T(5)+.3792410*T(6)-.5247277*T(7)          BESK 660
     3+.5575368*T(8)-.4262633*T(9)+.2184518*T(10)-.06680977*T(11)       BESK 670
     4+.009189383*T(12))*C                                              BESK 680
      IF(N)20,28,29                                                     BESK 690
   28 BK=G0                                                             BESK 700
      RETURN                                                            BESK 710
C                                                                       BESK 720
C     COMPUTE K1 USING POLYNOMIAL APPROXIMATION                         BESK 730
C                                                                       BESK 740
   29 G1=A*(1.2533141+.4699927*T(1)-.1468583*T(2)+.1280427*T(3)         BESK 750
     2-.1736432*T(4)+.2847618*T(5)-.4594342*T(6)+.6283381*T(7)          BESK 760
     3-.6632295*T(8)+.5050239*T(9)-.2581304*T(10)+.07880001*T(11)       BESK 770
     4-.01082418*T(12))*C                                               BESK 780
      IF(N-1)20,30,31                                                   BESK 790
   30 BK=G1                                                             BESK 800
      RETURN                                                            BESK 810
C                                                                       BESK 820
C     FROM K0,K1 COMPUTE KN USING RECURRENCE RELATION                   BESK 830
C                                                                       BESK 840
   31 DO 35 J=2,N                                                       BESK 850
      GJ=2.*(FLOAT(J)-1.)*G1/X+G0                                       BESK 860
      IF(GJ-1.0E70)33,33,32                                             BESK 870
   32 IER=4                                                             BESK 880
      GO TO 34                                                          BESK 890
   33 G0=G1                                                             BESK 900
   35 G1=GJ                                                             BESK 910
   34 BK=GJ                                                             BESK 920
      RETURN                                                            BESK 930
   36 B=X/2.                                                            BESK 940
```

TABLE 6.1.—*Listing of program for partial penetration in a leaky artesian aquifer*—Continued

```
      A=.5772157+ALOG(B)
      C=B*B
      IF(N-1)37,43,37
C
C     COMPUTE K0 USING SERIES EXPANSION
C
   37 G0=-A
      X2J=1.
      FACT=1.
      HJ=.0
      DO 40 J=1,6
      RJ=1./FLOAT(J)
      IF(X2J.LT.1.E-40) X2J=0.
C     PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW
C     PROBLEM ON WATFOR COMPILER
      X2J=X2J*C
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   40 G0=G0+X2J*FACT*(HJ-A)
      IF(N)43,42,43
   42 BK=G0
      RETURN
C
C     COMPUTE K1 USING SERIES EXPANSION
C
   43 X2J=B
      FACT=1.
      HJ=1.
      G1=1./X+X2J*(.5+A-HJ)
      DO 50 J=2,8
      X2J=X2J*C
      RJ=1./FLOAT(J)
      FACT=FACT*RJ*RJ
      HJ=HJ+RJ
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))
      IF(N-1)31,52,31
   52 BK=G1
      RETURN
      END
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*

```
C       ******************************************************************
C
C       PURPOSE
C          TO COMPUTE A TABLE OF FUNCTION VALUES FOR DRAWDOWN IN A
C          LEAKY ARTESIAN AQUIFER IN RESPONSE TO A STEP CHANGE IN
C          WATER LEVEL IN THE CONTROL WELL. FUNCTION VALUES ARE
C          EXPRESSED AS A FRACTION OF DRAWDOWN IN CONTROL WELL (S/SW).
C          REFERENCE - HANTUSH,M.S., 1959, NONSTEADY FLOW TO FLOWING
C          WELLS IN LEAKY AQUIFERS: JOUR. GEOPHYS. RESEARCH, V. 64,
C          NO. 8, P. 1043-1052.
C       INPUT DATA
C          1 CARD - FORMAT(2E10.5)
C                TSMALL - SMALLEST VALUE OF ALPHA FOR WHICH COMPUTATION
C                IS DESIRED.
C                TLARGE -  LARGEST VALUE OF ALPHA FOR WHICH COMPUTATION
C                IS DESIRED.
C          1 CARD - FORMAT(13F5.0)
C                BDAT - 13 VALUES OF RW/B. NON ZERO VALUES SHOULD BE GE 1
C                AND LT 10. FIRST ZERO (OR BLANK) WILL TERMINATE THE
C                LIST. AT LEAST ONE NON ZERO VALUE MUST BE CODED. INPUT
C                VALUES ARE MULTIPLIED BY POWER OF TEN DETERMINED BY
C                PROGRAM FROM ALPHA.
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
C        1 CARD = FORMAT(10F8.2)
C             RW = RADIUS OF CONTROL WELL.
C             RDAT = 9 VALUES OF RADIAL DISTANCE OF OBSERVATION POINTS
C             FROM CONTROL WELL. SHOULD BE CODED WITH SMALLEST NUMBER
C             FIRST, THEN BY INCREASING DISTANCE. THE FIRST ZERO
C             (OR BLANK) VALUE WILL TERMINATE COMPUTATION.
C     METHOD
C        EVALUATES EQ. 13 OF HANTUSH. EVALUATION OF BESSEL FUNCTIONS
C        BY SUBROUTINES BESK AND BESY AND FUNCTION J0. EVALUATES
C        INTEGRAL BY SUM, I=1 TO 8000, F((DELTA U)*(I-.5))*(DELTA U).
C        CHOOSES INITIAL DELTA U = .001/SQRT(SMALLEST ALPHA) AND USES
C        THIS VALUE FOR ALL RW/B GE 10*(DELTA U). FOR SMALLER RW/B,
C        DIVIDES DELTA U BY 10 AND MULTIPLIES SMALLEST ALPHA BY 100.
C     REMARKS
C        SMALLEST RW/B GE .01/SQRT(SMALLEST ALPHA)
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        BESK,BESY,J0
C
C     *****************************************************************
      REAL*8 SUM1,SUM2
      REAL*4 KOBP,KOB,J0,JOPU,JOU,Y(8000),J(8000),F(8000),FT(8000),
     1   FB(8000),RDAT(9),TDAT(6),BDAT(13),ARRAY(25,9,13),B(13),T(25)
      DATA FT/8000*0./,FB/8000*0./
      DATA RDAT/9*1./
      DATA ARRAY/2925*0./,TDAT/1.,1.5,2.,3.,5.,7./
      IRD=5
      IPT=6
      READ (IRD,24) TSMALL,TLARGE
      READ (IRD,23) BDAT
      READ (IRD,22) RW,RDAT
      IBEGIN=ALOG10(TSMALL)
      IEND=ALOG10(TLARGE)+.99999
      IF ((IBEGIN/2*2).LT.IBEGIN) IBEGIN=IBEGIN-1
      ISPAN=IEND-IBEGIN
      MLIMIT=(ISPAN+1)/2
C     COMPUTES INITIAL DELTA U (DU) = .001/SQRT(SMALLEST ALPHA)
      DU=.001/SQRT(TDAT(1)*10.**IBEGIN)
C     EXPONENT (JBEGIN) OF SMALLEST RW/B IS COMPUTED FROM EXPONENT
C     (IBEGIN) OF SMALLEST ALPHA.
      JBEGIN=-IBEGIN/2-2
      DO 1 I=1,13
      IF (BDAT(I).EQ.0.) GO TO 2
    1 CONTINUE
      NB=13
      GO TO 3
    2 NB=I-1
    3 CONTINUE
      DO 4 I=1,9
      IF (RDAT(I).EQ.0.) GO TO 5
    4 RDAT(I)=RDAT(I)/RW
      NR=9
      GO TO 6
    5 NR=I-1
    6 DO 21 M=1,MLIMIT
      NUM=8000
      START=-DU/2.
      U=START
      DO 7 I=1,NUM
      U=U+DU
      CALL BESY(U,0,Y(I),IDUMY)
    7 J(I)=J0(U)
      DO 19 IR=1,NR
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
      RHO=RDAT(IR)
      U=START
      DO 8 I=1,NUM
      U=U+DU
      CALL BESY(RHO*U,0,YOPU,IDUMY)
      JOPU=JO(RHO*U)
      JOU=J(I)
      YOU=Y(I)
    8 F(I)=(JOPU*YOU-YOPU*JOU)/(JOU*JOU+YOU*YOU)
      DO 19 IT=1,25
      INDEX=(IT-1)/6
      IEXP=IBEGIN+INDEX
      II=IT-INDEX*6
      TAU=TDAT(II)*10.**IEXP
      T(IT)=TAU
      U=START
      NUMT=NUM
      DO 9 I=1,NUMT
      U=U+DU
      FTEST=F(I)
      IF (ABS(FTEST).LT.1.E-30) GO TO 10
      XTEST=-TAU*U*U
      IF (XTEST+69.) 10,10,9
    9 FT(I)=FTEST*EXP(XTEST)
      GO TO 11
   10 NUMT=I-1
      FT(I)=0.
   11 DO 19 IB=1,13
      JNDEX=(IB-1)/NB
      JEXP=JBEGIN+JNDEX
      JJ=IB-JNDEX*NB
      BETA=BDAT(JJ)*10.**JEXP
      B(IB)=BETA
      U=START
      BSQ=BETA*BETA
      NUMB=NUMT
      DO 12 I=1,NUMB
      U=U+DU
      FTEST=FT(I)
      IF (ABS(FTEST).LT.1.E-30) GO TO 13
   12 FB(I)=FTEST/(U+BSQ/U)
      GO TO 14
   13 NUMB=I-1
      FB(I)=0.
   14 SUM1=0.
      SUM2=0.
      DO 15 I=1,NUMB,2
      SUM1=SUM1+FB(I)
   15 SUM2=SUM2+FB(I+1)
      XINT=(SUM1+SUM2)*DU
      CALL BESK(RHO*BETA,0,KOBP,IDUMY)
      CALL BESK(BETA,0,KOB,IDUMY)
      RATIO=0.
      IF (KOBP.GT.0.) RATIO=KOBP/KOB
      XTEST=-TAU*BSQ
      IF (XTEST+30.) 16,17,17
   16 XPT=0.
      GO TO 18
   17 XPT=EXP(XTEST)
   18 Z=RATIO+.6366198*XPT*XINT
      IF ((Z.LT.0.).AND.(Z.GT.-5.E-5)) Z=0.E0
   19 ARRAY(IT,IR,IB)=Z
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
      DO 20 K=1,NR
      WRITE (IPT,25) RDAT(K),B
      WRITE (IPT,26) (T(I),(ARRAY(I,K,L),L=1,13),I=1,25)
   20 CONTINUE
C     EXPONENT OF SMALLEST RW/B DECREASED BY ONE EACH TIME THROUGH LOOP
      JBEGIN=JBEGIN-1
C     EXPONENT OF SMALLEST ALPHA INCREASED BY TWO EACH TIME THROUGH LOOP
      IBEGIN=IBEGIN+2
C     DELTA U (DU) IS DIVIDED BY 10 EACH TIME THROUGH THE LOOP
   21 DU=.1*DU
      STOP
C
   22 FORMAT (10F8.2)
   23 FORMAT (13F5.0)
   24 FORMAT (2E10.5)
   25 FORMAT ('1','Z(ALPHA,R/RW,RW/B), R/RW=',F6.0/'0',9X,'| RW/B'/(' ',
     13X,'ALPHA |',13E9.2))
   26 FORMAT (' ',E10.3,13F9.3)
      END
      REAL FUNCTION J0*4(X)
C     ******************************************************************
C
C     FUNCTION J0
C
C     PURPOSE
C        TO COMPUTE THE ZERO ORDER J BESSEL FUNCTION FOR A GIVEN
C        ARGUMENT.
C     USAGE
C        J0(X)
C     DESCRIPTION OF PARAMETER
C        X - REAL*4, ARGUMENT OF J0 BESSEL FUNCTION DESIRED.
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE.
C     METHOD
C        POLYNOMIAL APPROXIMATION FOR X<4 AND ASYMPTOTIC SERIES FOR
C        X GE 4. THE POLYNOMIAL APPROXIMATION IS THE FIRST 10 TERMS OF
C        THE POWER SERIES FOR J0(X) (MILLER, K.S., 1957,
C        ENGINEERING MATHEMATICS: RINEHART AND CO., INC., NEW YORK,
C        P. 120). THE ASYMPTOTIC EXPANSION OF J0(X) IS GIVEN ON P. 82
C        OF BOWMAN, FRANK, 1958, INTRODUCTION  TO BESSEL FUNCTIONS:
C        DOVER PUBLICATIONS INC., NEW YORK. THE TERMS P ('A*P0') AND
C        Q ('-B*Q0') OF THE ASYMPTOTIC EXPANSION ARE COMPUTED BY AN
C        ALGORITHM FROM IBM SUBROUTINE BESY.
C
C     ******************************************************************
      IF (X-4.) 1,3,3
C     COMPUTE J0 BY FIRST 10 TERMS OF POWER SERIES
    1 A=-X*X/4.
      B=1.
      DO 2 I=1,10
      C=11.-I
    2 B=1.+B*(A/(C*C))
      J0=B
      GO TO 4
C     COMPUTE J0 BY ASYMPTOTIC SERIES
    3 T1=4./X
      T2=T1*T1
      P0=((((-.0000037043*T2+.0000173565)*T2-.0000487613)*T2+.00017343)*
     1T2-.001753062)*T2+.3989423
      Q0=((((.0000032312*T2-.0000142078)*T2+.0000342468)*T2-.0000869791)
     1*T2+.0004564324)*T2-.01246694
      A=2.0/SQRT(X)
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
      B=A*T1
      C=X-.7853982
      J0=A*P0*COS(C)-B*Q0*SIN(C)
    4 RETURN
      END
      SUBROUTINE BESY(X,N,BY,IER)
C
C     ..................................................................
C
C        SUBROUTINE BESY
C
C        PURPOSE
C           COMPUTE THE Y BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER
C
C        USAGE
C           CALL BESY(X,N,BY,IER)
C
C        DESCRIPTION OF PARAMETERS
C           X  -THE ARGUMENT OF THE Y BESSEL FUNCTION DESIRED
C           N  -THE ORDER OF THE Y BESSEL FUNCTION DESIRED
C           BY -THE RESULTANT Y BESSEL FUNCTION
C           IER-RESULTANT ERROR CODE WHERE
C              IER=0  NO ERROR
C              IER=1  N IS NEGATIVE
C              IER=2  X IS NEGATIVE OR ZERO
C              IER=3  BY HAS EXCEEDED MAGNITUDE OF 10**70
C
C        REMARKS
C           VERY SMALL VALUES OF X MAY CAUSE THE RANGE OF THE LIBRARY
C           FUNCTION ALOG TO BE EXCEEDED
C           X MUST BE GREATER THAN ZERO
C           N MUST BE GREATER THAN OR EQUAL TO ZERO
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE
C           AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED
C           FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE
C           UNIVERSITY PRESS, 1958, P. 62
C
C     ..................................................................
C
C
C     CHECK FOR ERRORS IN N AND X
C
      IF(N)180,10,10
   10 IER=0
      IF(X)190,190,20
C
C     BRANCH IF X LESS THAN OR EQUAL 4
C
   20 IF(X-4.0)40,40,30
C
C       COMPUTE Y0 AND Y1 FOR X GREATER THAN 4
C
   30 T1=4.0/X
      T2=T1*T1
      P0=((((-.0000037043*T2+.0000173565)*T2-.0000487613)*T2
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
     1  +.00017343)*T2-.001753062)*T2+.3989423
        Q0=((((.0000032312*T2-.0000142078)*T2+.0000342468)*T2
     1  -.0000869791)*T2+.0004564324)*T2-.01246694
        P1=((((.0000042414*T2-.0000200920)*T2+.0000580759)*T2
     1  -.000223203)*T2+.002921826)*T2+.3989423
        Q1=((((-.0000036594*T2+.00001622)*T2-.0000398708)*T2
     1  +.0001064741)*T2-.0006390400)*T2+.03740084
        A=2.0/SQRT(X)
        B=A*T1
        C=X-.7853982
        Y0=A*P0*SIN(C)+B*Q0*COS(C)
        Y1=-A*P1*COS(C)+B*Q1*SIN(C)
        GO TO 90
C
C          COMPUTE Y0 AND Y1 FOR X LESS THAN OR EQUAL TO 4
C
     40 XX=X/2.
        X2=XX*XX
        T=ALOG(XX)+.5772157
        SUM=0.
        TERM=T
        Y0=T
        DO 70 L=1,15
        IF(L-1)50,60,50
     50 SUM=SUM+1./FLOAT(L-1)
     60 FL=L
        TS=T-SUM
        IF(ABS(TERM).LE.1.E-40) TERM=0.
        TERM=(TERM*(-X2)/FL**2)*(1.-1./(FL*TS))
     70 Y0=Y0+TERM
        TERM = XX*(T-.5)
        SUM=0.
        Y1=TERM
        DO 80 L=2,16
        SUM=SUM+1./FLOAT(L-1)
        FL=L
        FL1=FL-1.
        TS=T-SUM
        IF(ABS(TERM).LE.1.E-40) TERM=0.
        TERM=(TERM*(-X2)/(FL1*FL))*((TS-.5/FL)/(TS+.5/FL1))
     80 Y1=Y1+TERM
        PI2=.6366198
        Y0=PI2*Y0
        Y1=-PI2/X+PI2*Y1
C
C       CHECK IF ONLY Y0 OR Y1 IS DESIRED
C
     90 IF(N-1)100,100,130
C
C       RETURN EITHER Y0 OR Y1 AS REQUIRED
C
    100 IF(N)110,120,110
    110 BY=Y1
        GO TO 170
    120 BY=Y0
        GO TO 170
C
C     PERFORM RECURRENCE OPERATIONS TO FIND YN(X)
C
    130 YA=Y0
        YB=Y1
        K=1
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
  140 T=FLOAT(2*K)/X                                                    BESY1180
      YC=T*YB-YA                                                        BESY1190
      IF(ABS(YC)-1.0E70)145,145,141                                     BESY1200
  141 IER=3                                                             BESY1210
      RETURN                                                            BESY1220
  145 K=K+1                                                             BESY1230
      IF(K-N)150,160,150                                                BESY1240
  150 YA=YB                                                             BESY1250
      YB=YC                                                             BESY1260
      GO TO 140                                                         BESY1270
  160 BY=YC                                                             BESY1280
  170 RETURN                                                            BESY1290
  180 IER=1                                                             BESY1300
      RETURN                                                            BESY1310
  190 IER=2                                                             BESY1320
      RETURN                                                            BESY1330
      END                                                               BESY1340
      SUBROUTINE BESK(X,N,BK,IER)                                       BESK 410
C                                                                       BESK  10
C     ..................................................................BESK  20
C                                                                       BESK  30
C        SUBROUTINE BESK                                                BESK  40
C                                                                       BESK  50
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDERBESK  60
C                                                                       BESK  70
C        USAGE                                                          BESK  80
C           CALL BESK(X,N,BK,IER)                                       BESK  90
C                                                                       BESK 100
C        DESCRIPTION OF PARAMETERS                                      BESK 110
C           X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED           BESK 120
C           N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED              BESK 130
C           BK -THE RESULTANT K BESSEL FUNCTION                         BESK 140
C           IER-RESULTANT ERROR CODE WHERE                              BESK 150
C              IER=0  NO ERROR                                          BESK 160
C              IER=1  N IS NEGATIVE                                     BESK 170
C              IER=2  X IS ZERO OR NEGATIVE                             BESK 180
C              IER=3  X .GT. 170, MACHINE RANGE EXCEEDED                BESK 190
C              IER=4  BK .GT. 10**70                                    BESK 200
C                                                                       BESK 210
C        REMARKS                                                        BESK 220
C           N MUST BE GREATER THAN OR EQUAL TO ZERO                     BESK 230
C                                                                       BESK 240
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  BESK 250
C           NONE                                                        BESK 260
C                                                                       BESK 270
C        METHOD                                                         BESK 280
C           COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING  BESK 290
C           SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION BESK 300
C           USING RECURRENCE RELATION.                                  BESK 310
C           RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE  BESK 320
C           AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS  BESK 330
C           TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED    BESK 340
C           FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,  BESK 350
C           'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE    BESK 360
C           UNIVERSITY PRESS, 1958, P. 62                               BESK 370
C                                                                       BESK 380
C     ..................................................................BESK 390
C                                                                       BESK 400
      DIMENSION T(12)                                                   BESK 420
      BK=.0                                                             BESK 430
      IF(N)10,11,11                                                     BESK 440
   10 IER=1                                                             BESK 450
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
      RETURN                                                            BESK 460
   11 IF(X)12,12,20                                                     BESK 470
   12 IER=2                                                             BESK 480
      RETURN                                                            BESK 490
   20 IF(X-170.0)22,22,21                                               BESK 500
   21 IER=3                                                             BESK 510
      RETURN                                                            BESK 520
   22 IER=0                                                             BESK 530
      IF(X-1.)36,36,25                                                  BESK 540
   25 A=EXP(-X)                                                         BESK 550
      B=1./X                                                            BESK 560
      C=SQRT(B)                                                         BESK 570
      T(1)=B                                                            BESK 580
      DO 26 L=2,12                                                      BESK 590
   26 T(L)=T(L-1)*B                                                     BESK 600
      IF(N-1)27,29,27                                                   BESK 610
C                                                                       BESK 620
C     COMPUTE K0 USING POLYNOMIAL APPROXIMATION                         BESK 630
C                                                                       BESK 640
   27 G0=A*(1.2533141-.1566642*T(1)+.08811128*T(2)-.09139095*T(3)       BESK 650
     2+.1344596*T(4)-.2299850*T(5)+.3792410*T(6)-.5247277*T(7)          BESK 660
     3+.5575368*T(8)-.4262633*T(9)+.2184518*T(10)-.06680977*T(11)       BESK 670
     4+.009189383*T(12))*C                                              BESK 680
      IF(N)20,28,29                                                     BESK 690
   28 BK=G0                                                             BESK 700
      RETURN                                                            BESK 710
C                                                                       BESK 720
C     COMPUTE K1 USING POLYNOMIAL APPROXIMATION                         BESK 730
C                                                                       BESK 740
   29 G1=A*(1.2533141+.4699927*T(1)-.1468583*T(2)+.1280427*T(3)         BESK 750
     2-.1736432*T(4)+.2847618*T(5)-.4594342*T(6)+.6283381*T(7)          BESK 760
     3-.6632295*T(8)+.5050239*T(9)-.2581304*T(10)+.07880001*T(11)       BESK 770
     4-.01082418*T(12))*C                                               BESK 780
      IF(N-1)20,30,31                                                   BESK 790
   30 BK=G1                                                             BESK 800
      RETURN                                                            BESK 810
C                                                                       BESK 820
C     FROM K0,K1 COMPUTE KN USING RECURRENCE RELATION                   BESK 830
C                                                                       BESK 840
   31 DO 35 J=2,N                                                       BESK 850
      GJ=2.*(FLOAT(J)-1.)*G1/X+G0                                       BESK 860
      IF(GJ-1.0E70)33,33,32                                             BESK 870
   32 IER=4                                                             BESK 880
      GO TO 34                                                          BESK 890
   33 G0=G1                                                             BESK 900
   35 G1=GJ                                                             BESK 910
   34 BK=GJ                                                             BESK 920
      RETURN                                                            BESK 930
   36 B=X/2.                                                            BESK 940
      A=.5772157+ALOG(B)                                                BESK 950
      C=B*B                                                             BESK 960
      IF(N-1)37,43,37                                                   BESK 970
C                                                                       BESK 980
C     COMPUTE K0 USING SERIES EXPANSION                                 BESK 990
C                                                                       BESK1000
   37 G0=-A                                                             BESK1010
      X2J=1.                                                            BESK1020
      FACT=1.                                                           BESK1030
      HJ=.0                                                             BESK1040
      DO 40 J=1,6                                                       BESK1050
      RJ=1./FLOAT(J)                                                    BESK1060
      IF(X2J.LT.1.E-40) X2J=0.                                          BESK1061
C     PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW   BESK1062
```

TABLE 7.2.—*Listing of program for constant drawdown in a well in an infinite leaky aquifer*—Continued

```
C     PROBLEM ON WATFOR COMPILER                                      BESK1063
      X2J=X2J*C                                                       BESK1070
      FACT=FACT*RJ*RJ                                                 BESK1080
      HJ=HJ+RJ                                                        BESK1090
   40 G0=G0+X2J*FACT*(HJ-A)                                           BESK1100
      IF(N)43,42,43                                                   BESK1110
   42 BK=G0                                                           BESK1120
      RETURN                                                          BESK1130
C                                                                     BESK1140
C     COMPUTE K1 USING SERIES EXPANSION                               BESK1150
C                                                                     BESK1160
   43 X2J=B                                                           BESK1170
      FACT=1.                                                         BESK1180
      HJ=1.                                                           BESK1190
      G1=1./X+X2J*(.5+A-HJ)                                           BESK1200
      DO 50 J=2,8                                                     BESK1210
      X2J=X2J*C                                                       BESK1220
      RJ=1./FLOAT(J)                                                  BESK1230
      FACT=FACT*RJ*RJ                                                 BESK1240
      HJ=HJ+RJ                                                        BESK1250
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))                             BESK1260
      IF(N-1)31,52,31                                                 BESK1270
   52 BK=G1                                                           BESK1280
      RETURN                                                          BESK1290
      END                                                             BESK130-
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*

```
C***********************************************************************FAR   1
C                                                                       FAR   2
C     PURPOSE                                                           FAR   3
C        COMPUTES FUNCTION VALUES OF F(U,ALPHA,RHO) FOR RHO > 1 -       FAR   4
C        PAPADOPULOS,I.S. AND COOPER,H.H.,JR., 1967, DRAWDOWN IN        FAR   5
C        A WELL OF LARGE DIAMETER; WATER RESOURCES RESEARCH, V. 3,      FAR   6
C        NO. 1, P. 241-244.                                             FAR   7
C        PROGRAM BY S.S.PAPADOPULOS.                                    FAR   8
C     INPUT DATA - ONE OR MORE GROUPS, EACH GROUP CODED AS FOLLOWS      FAR   9
C        1 CARD - FORMAT(2E10.5)                                        FAR  10
C             ALPHA  -      RW**2*S/RC**2 - RADIUS OF WELL (SCREEN      FAR  11
C                  OR OPEN BORE IN AQUIFER) SQUARED * STORAGE           FAR  12
C                  COEFFICIENT / RADIUS OF CASING (OVER INTERVAL OF     FAR  13
C                  WATER LEVEL CHANGE) SQUARED.                         FAR  14
C             RHO - R/RW - DISTANCE FROM PUMPED WELL / RADIUS OF        FAR  15
C                  WELL (SCREEN OR OPEN BORE IN AQUIFER), MUST BE       FAR  16
C                  GREATER THAN ONE.                                    FAR  17
C        1 CARD - FORMAT(16E5.0)                                        FAR  18
C             U- 16 VALUES OF U - R**2*S/(4*T*TIME) - DISTANCE FROM     FAR  19
C                  PUMPED WELL SQUARED * STORAGE COEFFICIENT /          FAR  20
C                  4 * TRANSMISSIVITY * TIME. IF LESS THAN 16 DESIRED,  FAR  21
C                  BLANK OR ZERO VALUES MAY BE CODED FOR THE REST.      FAR  22
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     FAR  23
C        PEAK,SIMP,APEKE,EXBSL1,JY0,JY1,ROOTS - MUST BE IN DECK.        FAR  24
C                                                                       FAR  25
C***********************************************************************FAR  26
      DIMENSION V(40,40),U(16)                                          FAR  27
      COMMON XPK,YPK                                                    FAR  28
      COMMON/PBLK/A,B,RHO                                               FAR  29
      EXTERNAL EXBSL1                                                   FAR  30
    1 READ (5,16,END=15) ALPHA,RHO                                      FAR  31
      IF (ALPHA) 15,15,2                                                FAR  32
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
    2 WRITE (6,17) ALPHA,RHO
      WRITE (6,18)
    3 READ (5,19) U
      DO 14 II=1,16
      IF (U(II)) 1,1,4
    4 A=ALPHA+ALPHA
      B=0.25/U(II)
      CALL APEKE(EXBSL1)
      CALL PEAK(EXBSL1)
      IF (XPK-1.0E-8) 5,6,6
    5 WRITE (6,20) XPK,U
      GO TO 3
    6 IF (XPK-3.0) 8,7,7
    7 WRITE (6,21) XPK,U
      GO TO 3
    8 EPS=0.000001
      HBAR=0.007*XPK
      CALL SIMPS(0.0,XPK,EPS,HBAR,SUM,DEL,EXBSL1)
      XM1=((3.14159265*7.0)/(8.0*(RHO-1.))+1.E-6)*RHO/2.
      DX1=XM1-(1.0E-6)*RHO
      DXN=(2.0*3.14159265*RHO)/(5.*(RHO-1.))
      DL=3.14159265*RHO/(RHO-1.)
      CALL ROOTS(XM1,DX1,RT1,EXBSL1)
      HBAR=0.007*(RT1-XPK)
      CALL SIMPS(XPK,RT1,EPS,HBAR,TRM1,ERR1,EXBSL1)
      SUM=SUM+TRM1
      DEL=DEL+ERR1
      X1=RT1
      I=1
    9 XM=X1+DL
      CALL ROOTS(XM,DXN,X2,EXBSL1)
      HBAR=0.007*(X2-X1)
      CALL SIMPS(X1,X2,EPS,HBAR,TRM,ERR,EXBSL1)
      V(1,I)=ABS(TRM)
      DEL=DEL+ERR
      I=I+1
      IF (I-40) 10,10,11
   10 X1=X2
      GO TO 9
   11 EST=0.0
      DO 12 K=2,40
      M=41-K
      DO 12 J=1,M
   12 V(K,J)=V(K-1,J+1)-V(K-1,J)
      DO 13 N=1,40
      L=N-1
      DELV=(-0.5)**L*V(N,1)
   13 EST=EST+(0.5)*DELV
      SUM=SUM-EST
      PUAR=4.0*A*RHO*SUM/3.14159265
      WRITE (6,22) U(II),SUM,DEL,PUAR
   14 CONTINUE
      GO TO 1
   15 STOP
C
   16 FORMAT (2E10.5)
   17 FORMAT ('1','F(U,ALPHA,RHO) FOR ALPHA=',1PE13.5,', RHO=',1E13.5)
   18 FORMAT (1H0,12X,1HU,16X,8HINTEGRAL,9X,14HINTEGRAL ERROR,6X,14HF(U,
     1ALPHA,RHO)/1H )
   19 FORMAT (16E5.0)
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
   20 FORMAT (5H XPK=,E15.8,3X,16HTOO SMALL FOR U=,E10.3)
   21 FORMAT (5H XPK=,E15.8,3X,16HTOO LARGE FOR U=,E10.3)
   22 FORMAT (1H ,1P4E20.8)
      END
      FUNCTION EXBSL1(X)
C***********************************************************************
C
C     PURPOSE
C        COMPUTES VALUES OF THE INTEGRAND FOR F(U,ALPHA,RHO)
C     DESCRIPTION OF PARAMETER
C        X- REAL - ARGUMENT OF INTEGRAND
C
C***********************************************************************
      COMMON/PBLK/A,B,R
      IF (X) 1,1,2
    1 EXBSL1=0.
      GO TO 8
    2 W=X/R
      IF (W-1.0E7) 4,4,3
    3 FNU=A*COS(W*(R-1.0))-W*SIN(W*(R-1.0))
      DE=(W*W*SQRT(R))*(W*W+A*A)
      EXBSL1=FNU/DE
      GO TO 8
    4 Y=B*X*X
      IF (Y-0.01) 5,5,6
    5 EXPO=Y*(1.0-Y*(0.5-Y*((1.0/6.0)-Y*(1.0/24.0))))
      GO TO 7
    6 EXPO=1.0-EXP(-Y)
    7 CALL JY0(W,WJ0,WY0)
      CALL JY1(W,WJ1,WY1)
      AW=W*WY0-A*WY1
      BW=W*WJ0-A*WJ1
      CALL JY0(X,BJ0,BY0)
      FNUM=EXPO*(AW*BJ0-BW*BY0)
      DEN=X*X*(AW*AW+BW*BW)
      EXBSL1=FNUM/DEN
    8 RETURN
      END
      SUBROUTINE ROOTS(XM,DX,ROOT,F)
C***********************************************************************
C
C     PURPOSE
C        SEARCHES FOR ROOT OF F IN THE INTERVAL XM-DX TO XM+DX.
C     DESCRIPTION OF PARAMETERS - ALL REAL
C        XM - CENTER OF INTERVAL SEARCHED.
C        DX - HALF WIDTH OF INTERVAL SEARCHED.
C        ROOT - RETURNED ROOT LOCATION.
C        F - FUNCTION REFERENCE.
C
C***********************************************************************
      XL=XM-DX
      XR=XM+DX
      YL=F(XL)
      YR=F(XR)
      EP=0.000001*ABS(YL)
      DO 9 I=1,200
      YM=F(XM)
      UP=ABS(YM)
      IF (UP.LT.EP.AND.UP.LT.1.0D-7) GO TO 1
      IF (YM) 2,1,2
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
    1 ROOT=XM
      GO TO 10
    2 IF (YM*YL) 7,3,4
    3 ROOT=XL
      GO TO 10
    4 IF (YM*YR) 8,5,6
    5 ROOT=XR
      GO TO 10
    6 WRITE (6,11) XL,XR
      STOP
    7 XR=XM
      YR=YM
      GO TO 9
    8 XL=XM
      YL=YM
    9 XM=(XL+XR)/2.0
      ROOT=XM
   10 RETURN
C
   11 FORMAT (1H ,10X,27HNO ROOT IN INTERVAL XM-DX =,1PE20.8,5X,11HAND X
     1M+DX =,1PE20.8/)
      END
      SUBROUTINE APEKE(EXBSL)
C***********************************************************************
C
C     PURPOSE
C        GETS FIRST APPROXIMATION TO PEAK POSITION
C
C***********************************************************************
      COMMON XPK,YPK
      XPK=0.0
      YPK=0.0
      DO 2 I=1,17
      X=10.0**(I-9)
      Y=EXBSL(X)
      IF (Y-YPK) 3,3,1
    1 XPK=X
      YPK=Y
    2 CONTINUE
    3 RETURN
      END
      SUBROUTINE PEAK(EXBSL)
C***********************************************************************
C
C     PURPOSE
C        ATTEMPTS TO FIND POSITION OF MAXIMUM FOR INTEGRAND
C
C***********************************************************************
      COMMON XPK,YPK
      YPK=EXBSL(XPK)
      DO 13 L=1,200
      DX=0.01*XPK
      XL=XPK-DX
      YL=EXBSL(XL)
      XR=XPK+DX
      YR=EXBSL(XR)
      DEN=YR+YL-YPK-YPK
      IF (DEN) 1,9,1
    1 X=XPK-0.5*(YR-YL)*DX/DEN
    2 IF (X) 3,4,4
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
    3 X=0.0
    4 Y=EXBSL(X)
      IF (YR-Y) 6,6,5
    5 Y=YR
      X=XR
    6 IF (YL-Y) 8,8,7
    7 Y=YL
      X=XL
    8 IF (Y-YPK) 14,14,12
    9 IF (YR-YPK) 11,10,10
   10 X=XPK+DX+DX
      GO TO 2
   11 X=XPK-DX-DX
      GO TO 2
   12 YPK=Y
      XPK=X
   13 CONTINUE
   14 RETURN
      END

      SUBROUTINE SIMPS(Q,R,EPS,HBAR,AREA,DEL,F)
C***********************************************************************
C
C     PURPOSE
C         TO DETERMINE THE INTEGRAL OF A FUNCTION, F, FROM Q TO R,
C         USING SIMPSON'S RULE.
C     DESCRIPTION OF PARAMETERS
C         ALL REAL
C         Q - LOWER LIMIT OF INTEGRAL
C         R - UPPER LIMIT OF INTEGRAL
C         EPS - DESIRED ACCURACY
C         HBAR - MINIMUM DIVISION OF THE INTERVAL
C         AREA - COMPUTED VALUE OF INTEGRAL BETWEEN Q AND R
C         DEL - COMPUTED ESTIMATE OF ERROR
C         F- THE INTEGRAND (FUNCTION REFERENCE)
C     METHOD
C         USES SIMPSON'S RULE TO COMPUTE A SUM APPROXIMATING THE INTEGRAL
C         USES INITIAL H=(R-Q)/2. COMPUTES A SEQUENCE OF SUMS BY HALVING
C         H EACH TIME. COMPUTES ESTIMATE OF ERROR (DEL) AS (PREVIOUS
C         SUM - CURRENT SUM)/15. COMPUTATION STOPS WHEN 1) H<HBAR,
C         2) ABS(DEL)<ABS(EPS*CURRENT SUM). IF HBAR IS LE 0,
C         THEN HBAR=.007*(R-Q).
C
C***********************************************************************
      H=R-Q
      IF (H) 1,1,2
    1 AREA=0.0
      DEL=0.0
      GO TO 10
C R MUST BE GREATER THAN Q
    2 SP=1.0E35
      S3=0.0
      S1=F(Q)+F(R)
      IF (HBAR) 3,3,4
    3 HBAR=0.007*H
    4 S2=0.0
      X=Q+0.5*H
    5 S2=S2+4.0*F(X)
      X=X+H
      IF (X-R) 5,5,6
    6 SC=(S1+S2+S3)*H*0.16666667
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
      DEL=0.066666667*(SP-SC)                                           SIM  42
      IF (ABS(DEL)-ABS(EPS*SC)) 7,8,8                                   SIM  43
    7 AREA=SC-DEL                                                       SIM  44
      GO TO 10                                                          SIM  45
    8 S3=S3+0.5*S2                                                      SIM  46
      H=0.5*H                                                           SIM  47
      IF (H-HBAR) 7,9,9                                                 SIM  48
    9 SP=SC                                                             SIM  49
      GO TO 4                                                           SIM  50
   10 RETURN                                                            SIM  51
      END                                                               SIM  52-
      SUBROUTINE JY0(X,J0,Y0)                                           JY0   1
C***********************************************************************JY0   2
C                                                                       JY0   3
C     PURPOSE                                                           JY0   4
C        COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,        JY0   5
C        ZERO ORDER, FOR POSITIVE ARGUMENTS.                            JY0   6
C        SEE NBS AMS 55, P. 369-370.                                    JY0   7
C     DESCRIPTION OF PARAMETERS - ALL REAL                              JY0   8
C        X- ARGUMENT, MUST BE >0                                        JY0   9
C        J0 - RETURNED FUNCTION VALUE, J0(X)                            JY0  10
C        Y0 - RETURNED FUNCTION VALUE, Y0(X)                            JY0  11
C                                                                       JY0  12
C***********************************************************************JY0  13
      REAL J0                                                           JY0  14
      IF (X-3.0) 1,2,3                                                  JY0  15
    1 IF (X) 4,4,2                                                      JY0  16
    2 Z=(0.33333333*X)**2                                               JY0  17
      J0=1.0-Z*(2.2499997-Z*(1.2656208-Z*(0.3163866-Z*(0.0444479-Z*(0.00JY0  18
     139444-0.00021*Z)))))                                              JY0  19
      Y0=0.63661977*ALOG(0.5*X)*J0+0.36746691+Z*(0.60559366-Z*(0.7435038JY0  20
     14-Z*(0.25300117-Z*(0.04261214-Z*(0.00427916-0.00024846*Z)))))     JY0  21
      RETURN                                                            JY0  22
    3 Z=3.0/X                                                           JY0  23
      F=0.79788456-Z*(0.77E-6+Z*(0.0055274+Z*(0.00009512-Z*(0.00137237-ZJY0  24
     1*(0.00072805-0.00014476*Z)))))                                    JY0  25
      P=0.78539816+Z*(0.04166397+Z*(0.00003954-Z*(0.00262573-Z*(0.000541JY0  26
     125+Z*(0.00029333-0.00013558*Z)))))                                JY0  27
      Q=SQRT(1.0/X)                                                     JY0  28
      J0=Q*F*COS(X-P)                                                   JY0  29
      Y0=Q*F*SIN(X-P)                                                   JY0  30
    4 RETURN                                                            JY0  31
      END                                                               JY0  32-
      SUBROUTINE JY1(X,J1,Y1)                                           JY1   1
C***********************************************************************JY1   2
C                                                                       JY1   3
C     PURPOSE                                                           JY1   4
C        COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,        JY1   5
C        FIRST ORDER, FOR POSITIVE ARGUMENTS.                           JY1   6
C        SEE NBS AMS 55, P. 370.                                        JY1   7
C     DESCRIPTION OF PARAMETERS - ALL REAL                              JY1   8
C        X- ARGUMENT, MUST BE >0                                        JY1   9
C        J1 - RETURNED FUNCTION VALUE, J1(X)                            JY1  10
C        Y1 - RETURNED FUNCTION VALUE, Y1(X)                            JY1  11
C                                                                       JY1  12
C***********************************************************************JY1  13
      REAL J1                                                           JY1  14
      IF (X-3.0) 1,2,3                                                  JY1  15
    1 IF (X) 4,4,2                                                      JY1  16
    2 Z=(0.33333333*X)**2                                               JY1  17
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
      J1=X*(0.5-Z*(0.56249985-Z*(0.21093573-Z*(0.03954289-Z*(0.00443319-JY1  18
     1Z*(0.00031761-0.00001109*Z))))))                                 JY1  19
      Y1=0.63661977*ALOG(0.5*X)*J1+(-0.6366198+Z*(0.2212091+Z*(2.1682709JY1  20
     1-Z*(1.3164827-Z*(0.3123951-Z*(0.0400976-0.0027873*Z))))))/X      JY1  21
      RETURN                                                           JY1  22
    3 Z=3.0/X                                                          JY1  23
      F=0.79788456+Z*(0.156E-5+Z*(0.01659667+Z*(0.00017105-Z*(0.00249511JY1  24
     1-Z*(0.00113653-0.00020033*Z)))))                                 JY1  25
      P=0.78539816-Z*(0.12499612+Z*(0.0000565-Z*(0.00637879-Z*(0.0007434JY1  26
     18+Z*(0.00079824-0.00029166*Z)))))                                JY1  27
      Q=SQRT(1.0/X)                                                    JY1  28
      J1=Q*F*SIN(X-P)                                                  JY1  29
      Y1=-Q*F*COS(X-P)                                                 JY1  30
    4 RETURN                                                           JY1  31
      END                                                              JY1  32-
C**********************************************************************FUA   1
C                                                                      FUA   2
C     PURPOSE                                                          FUA   3
C        COMPUTES FUNCTION VALUES OF F(UW,ALPHA)  -                    FUA   4
C        PAPADOPULOS,I.S. AND COOPER,H.H.,JR., 1967, DRAWDOWN IN       FUA   5
C        A WELL OF LARGE DIAMETER: WATER RESOURCES RESEARCH, V. 3,     FUA   6
C        NO. 1, P. 241-244.                                            FUA   7
C        PROGRAM BY S.S.PAPADOPULOS.                                   FUA   8
C     INPUT DATA - ONE OR MORE GROUPS, EACH GROUP CODED AS FOLLOWS     FUA   9
C        1 CARD - FORMAT (E10.5)                                       FUA  10
C             S - (ALPHA) - RW**2*S/RC**2 - RADIUS OF WELL (SCREEN     FUA  11
C                  OR OPEN BORE IN AQUIFER) SQUARED * STORAGE          FUA  12
C                  COEFFICIENT / RADIUS OF CASING (OVER INTERVAL OF    FUA  13
C                  WATER LEVEL CHANGE) SQUARED.                        FUA  14
C        1 CARD - FORMAT(16E5.0)                                       FUA  15
C             U- 16 VALUES OF UW - RW**2*S/(4*T*TIME) - RADIUS OF      FUA  16
C                  PUMPED WELL SQUARED * STORAGE COEFFICIENT /         FUA  17
C                  4 * TRANSMISSIVITY * TIME. IF LESS THAN 16 DESIRED, FUA  18
C                  BLANK OR ZERO VALUES MAY BE CODED FOR THE REST.     FUA  19
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                    FUA  20
C        PEAK,SIMP,APEKE,EXBSL2,JY0,JY1 - MUST BE INCLUDED IN DECK.    FUA  21
C                                                                      FUA  22
C**********************************************************************FUA  23
      COMMON XPK,YPK                                                   FUA  24
      COMMON/PBLK/A,B                                                  FUA  25
      EXTERNAL EXBSL2                                                  FUA  26
      DIMENSION U(16)                                                  FUA  27
      EPS=0.0001                                                       FUA  28
    1 READ (5,13,END=12) S                                             FUA  29
      IF (S) 1,1,2                                                     FUA  30
    2 READ (5,14) U                                                    FUA  31
      WRITE (6,15) S                                                   FUA  32
      DO 11 I=1,16                                                     FUA  33
      UW=U(I)                                                          FUA  34
      IF (UW) 1,1,3                                                    FUA  35
    3 B=0.25/UW                                                        FUA  36
      A=S+S                                                            FUA  37
      CALL APEKE(EXBSL2)                                               FUA  38
      CALL PEAK(EXBSL2)                                                FUA  39
      IF (XPK-1.0E-8) 4,5,5                                            FUA  40
    4 WRITE (6,16) UW,S,XPK,YPK                                        FUA  41
      GO TO 11                                                         FUA  42
    5 IF (XPK-1.0E8) 7,7,6                                             FUA  43
    6 WRITE (6,17) UW,S,XPK,YPK                                        FUA  44
      GO TO 11                                                         FUA  45
    7 HBAR=0.007*XPK                                                   FUA  46
```

TABLE 8.2.—*Listing of programs for constant discharge from a fully penetrating well of finite diameter*—Continued

```
      CALL SIMPS(0.0,XPK,EPS,HBAR,SUM,DEL,EXBSL2)
      X2=XPK
      DX=XPK
    8 DX=10.0*DX
      X1=X2
      X2=X1+DX
      Y=EXBSL2(X2)
      HBAR=0.007*DX
      CALL SIMPS(X1,X2,EPS,HBAR,TRM,ERR,EXBSL2)
      SUM=SUM+TRM
      DEL=DEL+ERR
      IF (X2-1.0E9) 9,10,10
    9 YT=1.5707963/X2**4
      IF (ABS(Y-YT)/YT-0.5E-6) 10,8,8
   10 EST=0.52359878/X2**3
      SUM=SUM+EST
      FUWS=3.2422779*S*S*SUM
      WRITE (6,18) UW,SUM,DEL,FUWS,XPK,YPK
   11 CONTINUE
      GO TO 1
   12 STOP
C
   13 FORMAT (E10.5)
   14 FORMAT (16E5.0)
   15 FORMAT ('1','F(UW,ALPHA) FOR ALPHA=',1PE14.5/'0',7X,'UW',12X,'INTE
     1GRAL',5X,'INTEGRAL ERROR',5X,'F(UW,ALPHA)',8X,'X(PEAK)',10X,'Y(PEA
     2K)'/' ')
   16 FORMAT (1H ,1PE14.7,9X,34HVALUES OF DUMMY VARIABLE TOO SMALL,1PE25
     1.7,1PE17.7)
   17 FORMAT (1H ,1PE14.7,9X,34HVALUES OF DUMMY VARIABLE TOO LARGE,1PE25
     1.7,1PE17.7)
   18 FORMAT (1H ,1PE14.5,1P5E17.5)
      END

      FUNCTION EXBSL2(X)
C***********************************************************************
C
C     PURPOSE
C        COMPUTES VALUES OF THE INTEGRAND FOR F(UW,ALPHA)
C     DESCRIPTION OF PARAMETER
C        X- REAL - ARGUMENT OF INTEGRAND
C
C***********************************************************************
      COMMON/PBLK/A,B
      IF (X) 1,1,2
    1 EXBSL2=0.
      GO TO 8
    2 IF (X-1.E+7) 4,4,3
    3 EXBSL2=1.5707963/X**4
      GO TO 8
    4 Y=B*X*X
      IF (Y-.01) 5,5,6
    5 FNUM=Y*(1.-Y*(.5-Y*((1./6.)-Y*(1./24.))))
      GO TO 7
    6 FNUM=1.-EXP(-Y)
    7 CALL JY0(X,BJ0,BY0)
      CALL JY1(X,BJ1,BY1)
      DEN=((X*BJ0-A*BJ1)**2+(X*BY0-A*BY1)**2)*X**3
      EXBSL2=FNUM/DEN
    8 RETURN
      END
```

TABLE 9.2.—*Listing of program to compute change in water level due to sudden injection of a slug of water into a well*

```
C***********************************************************************
C
C     PURPOSE
C        COMPUTES FUNCTION VALUES OF F(BETA,ALPHA) = THE SLUG TEST
C        FUNCTION = COOPER,H.H.,JR.,  BREDEHOEFT,J.D., AND PAPADOPULOS,
C        I.S., 1967, RESPONSE OF A FINITE=DIAMETER WELL TO AN
C        INSTANTANEOUS CHARGE OF WATER: WATER RESOURCES RESEARCH,
C        V. 3, NO. 1, P. 263=269.
C        PROGRAM BY S.S.PAPADOPULOS.
C     INPUT DATA
C        1 OR MORE CARDS = FORMAT(F16.5)
C             A = (ALPHA) = RW**2*S/RC**2 = RADIUS OF WELL (SCREEN OR
C                  OPEN BORE IN AQUIFER) SQUARED * STORAGE COEFFICIENT
C                  / RADIUS OF CASING (OVER INTERVAL OF WATER LEVEL
C                  CHANGE) SQUARED.
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        PRX,DJY0,DJY1,DSIMPS = MUST BE INCLUDED IN DECK
C     METHOD
C        THIS PROGRAM CALCULATES THE SLUG TEST FUNCTION, F(BETA,ALPHA),
C        FOR VALUES OF BETA RANGING FROM 0.001 TO 1000.0 BY INCREMENTING
C        BETA ACCORDING TO DATA ARRAY BB(I). AVERAGE COMPUTATION TIME
C        IS ABOUT 30 SECONDS PER VALUE OF ALPHA ON IBM 360/155.
C
C***********************************************************************
      DOUBLE PRECISION A,B,PI,ZZ,EPS,Y,X1,X2,TERM,FAB,DATAN,DEL,HBAR
      DIMENSION ZZ(40), BB(39)
      COMMON A,B,PI
      EXTERNAL PRX
      DATA ZZ/0.D+0,1.D=10,1.D=9,1.D=8,1.D=7,1.D=6,1.D=5,1.D=4,
     1   1.D=3,1.D=2,1.D=1,2.D=1,3.D=1,4.D=1,5.D=1,6.D=1,7.D=1,8.D=1,
     2   9.D=1,1.D+0,2.D+0,3.D+0,4.D+0,5.D+0,6.D+0,7.D+0,8.D+0,
     3   9.D+0,1.D+1,2.D+1,3.D+1,4.D+1,5.D+1,6.D+1,7.D+1,8.D+1,
     4   9.D+1,1.D+2,1.25D+2,1.5D+2/
      DATA BB/.001,.002,.004,.006,.008,.01,.02,.04,.06,.08,.1,.2,.4,.6,.
     18,1.,2.,3.,4.,5.,6.,7.,8.,9.,10.,20.,30.,40.,50.,60.,70.,80.,90.,1
     200.,200.,400.,600.,800.,1000./
      PI=4.*DATAN(1.0D+00)
      EPS=0.00001
    1 READ (5,6) A
      IF (A.LE.0.0) GO TO 5
      WRITE (6,7) A
      WRITE (6,8)
      DO 4 I=1,39
      B=BB(I)
      Y=0.0
      DO 2 L=1,39
      X1=ZZ(L)
      X2=ZZ(L+1)
      HBAR=0.
      CALL DSIMPS(X1,X2,EPS,HBAR,TERM,DEL,PRX)
      Y=Y+TERM
      IF (L.GT.20.AND.TERM.LT.EPS) GO TO 3
    2 CONTINUE
    3 FAB=4.*A*Y/(PI*PI)
    4 WRITE (6,9) B,FAB
      GO TO 1
    5 STOP
C
C
    6 FORMAT (F16.5)
    7 FORMAT ('1',41X,'F(BETA,ALPHA) FOR ALPHA=',1PD9.2/)
    8 FORMAT ('0',53X,'BETA',13X,'H/H0'/)
    9 FORMAT (' ',51X,1PD8.2,10X,0PF6.4)
      END
```

TABLE 9.2.—*Listing of program to compute change in water level due to sudden injection of a slug of water into a well*—Continued

```
      DOUBLE PRECISION FUNCTION PRX(X)
C**********************************************************************
C
C     PURPOSE
C        COMPUTE VALUES OF THE INTEGRAND FOR F(BETA,ALPHA)
C     DESCRIPTION OF PARAMETER
C        X = DOUBLE PRECISION - ARGUMENT OF INTEGRAND
C
C**********************************************************************
      DOUBLE PRECISION A,B,PI,XX,X,C,F1,F2,J0,Y0,J1,Y1
      DOUBLE PRECISION DLOG,DSQRT,DEXP
      COMMON A,B,PI
      XX=DSQRT(A*X/B)
      IF (X) 6,1,2
    1 PRX=(PI*PI)/(16.*A*B)
      GO TO 6
    2 IF (X.LT.150.) GO TO 3
      PRX=0.0
      GO TO 6
    3 IF (XX.GT.0.0001) GO TO 4
      C=DEXP(5.772156649D-01)/2.
      F1=PI*X*(1.-A)
      F2=X*DLOG(C*C*A*X/B)+4.*B
      PRX=(B*PI*PI*DEXP(-X))/(A*(F1*F1+F2*F2))
      GO TO 6
    4 IF (XX.LT.50.) GO TO 5
      PRX=(PI*DEXP(-X))/(2.*XX*(X+4.*A*B))
      GO TO 6
    5 CALL DJY0(XX,J0,Y0)
      CALL DJY1(XX,J1,Y1)
      F1=(XX*J0-2.*A*J1)
      F2=(XX*Y0-2.*A*Y1)
      PRX=DEXP(-X)/(X*(F1*F1+F2*F2))
    6 RETURN
      END

      SUBROUTINE DJY0(X,J0,Y0)
C***********************************************************************
C
C     PURPOSE
C        COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,
C        ZERO ORDER, FOR POSITIVE ARGUMENTS.
C     DESCRIPTION OF PARAMETERS - ALL DOUBLE PRECISION
C        X- ARGUMENT, MUST BE >0
C        J0 - RETURNED FUNCTION VALUE, J0(X)
C        Y0 - RETURNED FUNCTION VALUE, Y0(X)
C
C***********************************************************************
      DOUBLE PRECISION Z,J0,Y0,F,P,Q,U,W,X,DLOG,DCOS,DSIN,DSQRT
      IF (X-3.0) 1,2,3
    1 IF (X) 4,4,2
    2 Z=(X/3.0)**2
      J0=1.0-Z*(2.2499997-Z*(1.2656208-Z*(0.3163866-Z*(0.0444479-Z*(0.00
     139444-0.00021*Z)))))
      W=(0.5D0)*X
      Y0=0.63661977*DLOG(W)*J0+0.36746691+Z*(0.60559366-Z*(0.74350384-Z*
     1(0.25300117-Z*(0.04261214-Z*(0.00427916-0.00024846*Z)))))
      RETURN
    3 Z=3.0/X
      F=0.79788456-Z*(0.77D-6+Z*(0.0055274+Z*(0.00009512-Z*(0.00137237-Z
     1*(0.00072805-0.00014476*Z)))))
      P=0.78539816+Z*(0.04166397+Z*(0.00003954-Z*(0.00262573-Z*(0.000541
     125+Z*(0.00029333-0.00013558*Z)))))
```

TABLE 9.2.—*Listing of program to compute change in water level due to sudden injection of a slug of water into a well*—Continued

```
      U=(1.0D0)/X
      Q=DSQRT(U)
      J0=Q*F*DCOS(X-P)
      Y0=Q*F*DSIN(X-P)
    4 RETURN
      END
      SUBROUTINE DJY1(X,J1,Y1)
C***********************************************************************
C
C     PURPOSE
C        COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,
C        FIRST ORDER, FOR POSITIVE ARGUMENTS.
C     DESCRIPTION OF PARAMETERS - ALL DOUBLE PRECISION
C        X= ARGUMENT, MUST BE >0
C        J1 - RETURNED FUNCTION VALUE, J1(X)
C        Y1 - RETURNED FUNCTION VALUE, Y1(X)
C
C***********************************************************************
      DOUBLE PRECISION X,J1,Y1,Z,W,DLOG,F,P,U,Q,DSQRT,DSIN,DCOS
      IF (X-3.0) 1,2,3
    1 IF (X) 4,4,2
    2 Z=(X/3.0)**2
      J1=X*(0.5-Z*(0.56249985-Z*(0.21093573-Z*(0.03954289-Z*(0.00443319-
     1Z*(0.00031761-0.00001109*Z))))))
      W=(0.5D0)*X
      Y1=0.63661977*DLOG(W)*J1+(-0.6366198+Z*(0.2212091+Z*(2.1682709-Z*(
     11.3164827-Z*(0.3123951-Z*(0.0400976-0.0027873*Z))))))/X
      RETURN
    3 Z=3.0/X
      F=0.79788456+Z*(0.156D-5+Z*(0.01659667+Z*(0.00017105-Z*(0.00249511
     1-Z*(0.00113653-0.00020033*Z)))))
      P=0.78539816-Z*(0.12499612+Z*(0.0000565-Z*(0.00637879-Z*(0.0007434
     18+Z*(0.00079824-0.00029166*Z)))))
      U=(1.0D0)/X
      Q=DSQRT(U)
      J1=Q*F*DSIN(X-P)
      Y1=-Q*F*DCOS(X-P)
    4 RETURN
      END
      SUBROUTINE DSIMPS(A,B,EPS,HBAR,AREA,DEL,F)
C***********************************************************************
C
C     PURPOSE
C        TO DETERMINE THE INTEGRAL OF A FUNCTION, F, FROM A TO B,
C        USING SIMPSON'S RULE.
C     DESCRIPTION OF PARAMETERS
C        ALL DOUBLE PRECISION
C        A - LOWER LIMIT OF INTEGRAL
C        B - UPPER LIMIT OF INTEGRAL
C        EPS - DESIRED ACCURACY
C        HBAR - MINIMUM DIVISION OF THE INTERVAL
C        AREA - COMPUTED VALUE OF INTEGRAL BETWEEN Q AND R
C        DEL - COMPUTED ESTIMATE OF ERROR
C        F= THE INTEGRAND (FUNCTION REFERENCE)
C     METHOD
C        USES SIMPSON'S RULE TO COMPUTE A SUM APPROXIMATING THE INTEGRAL
C        USES INITIAL H=(B-A)/2. COMPUTES A SEQUENCE OF SUMS BY HALVING
C        H EACH TIME. COMPUTES ESTIMATE OF ERROR (DEL) AS (PREVIOUS
C        SUM - CURRENT SUM)/15. COMPUTATION STOPS WHEN 1) H<HBAR,
C        2) ABS(DEL)<ABS(EPS*CURRENT SUM). IF HBAR IS LE 0,
C        THEN HBAR=.007*(B-A).
C
```

TABLE 9.2.—*Listing of program to compute change in water level due to sudden injection of a slug of water into a well*—Continued

```
C************************************************************************DSI  24
      DOUBLE PRECISION H,HBAR,AREA,DEL,S1,S2,S3,SC,SP,X,A,B,EPS,F,DABS  DSI  25
C  AREA OF F FROM A TO B,EPS IS DESIRED ACCURACY, HBAR THE MINIMUM      DSI  26
C  ALLOWABLE INTERVAL, DEL THE ESTIMATE OF THE ERROR                    DSI  27
      H=B-A                                                             DSI  28
      IF (H) 1,1,2                                                      DSI  29
    1 AREA=0.0                                                          DSI  30
      DEL=0.0                                                           DSI  31
      GO TO 10                                                          DSI  32
    2 SP=1.0D35                                                         DSI  33
      S3=0.0                                                            DSI  34
      S1=F(A)+F(B)                                                      DSI  35
      IF (HBAR) 3,3,4                                                   DSI  36
    3 HBAR=0.007*H                                                      DSI  37
    4 S2=0.0                                                            DSI  38
      X=A+0.5*H                                                         DSI  39
    5 S2=S2+4.0*F(X)                                                    DSI  40
      X=X+H                                                             DSI  41
      IF (X-B) 5,5,6                                                    DSI  42
    6 SC=(S1+S2+S3)*H*0.1666666667                                      DSI  43
      DEL=0.06666666667*(SP-SC)                                         DSI  44
      IF (DABS(DEL)-DABS(EPS*SC)) 7,8,8                                 DSI  45
    7 AREA=SC-DEL                                                       DSI  46
      GO TO 10                                                          DSI  47
    8 S3=S3+0.5*S2                                                      DSI  48
      H=0.5*H                                                           DSI  49
      IF (H-HBAR) 7,9,9                                                 DSI  50
    9 SP=SC                                                             DSI  51
      GO TO 4                                                           DSI  52
   10 RETURN                                                            DSI  53
      END                                                               DSI  54-
```

TABLE 11.1.—*Listing of program to compute the convolution integral for a leaky aquifer*

```
C*********************************************************************** HRT   1
C                                                                       HRT   2
C     PURPOSE                                                           HRT   3
C        COMPUTES CHANGES IN WATER LEVEL, H(R,T), IN RESPONSE TO        HRT   4
C        VARYING DISCHARGE USING THE CONVOLUTION INTEGRAL FOR           HRT   5
C        LEAKY AQUIFERS - EQ. 3 OF MOENCH,ALLEN,1971, GROUND-WATER      HRT   6
C        FLUCTUATIONS IN RESPONSE TO ARBITRARY PUMPAGE: GROUND WATER,   HRT   7
C        V.9, NO.2,P.4-8.                                               HRT   8
C     INPUT DATA - ONE OR MORE GROUPS, EACH GROUP CODED AS FOLLOWS      HRT   9
C        1 CARD - FORMAT(2E10.5,4X,I1,5X,E10.5)                         HRT  10
C              TBEGIN - SMALLEST VALUE OF TIME FOR OUTPUT.              HRT  11
C              TEND - LARGEST VALUE OF TIME FOR OUTPUT.                 HRT  12
C              IQ - INDICATES FORM OF DISCHARGE FUNCTION, Q(T).         HRT  13
C                   IQ=1,2,3 REFER TO DISCHARGE FUNCTIONS IN            HRT  14
C                   HANTUSH,M.S., 1964, HYDRAULICS OF WELLS IN CHOW,    HRT  15
C                   VEN TE, ED., ADVANCES IN HYDROSCIENCE, VOL. 1:      HRT  16
C                   ACADEMIC PRESS INC., NEW YORK, P. 281-442.          HRT  17
C                   IQ=1, Q(T) IS AN EXPONENTIAL FUNCTION, CASE A,      HRT  18
C                         P. 343 OF HANTUSH.                            HRT  19
C                   IQ=2, Q(T) IS A HYPERBOLIC FUNCTION, CASE B,        HRT  20
C                         P. 344 OF HANTUSH.                            HRT  21
C                   IQ=3, Q(T) IS AN INVERSE SQUARE ROOT FUNCTION,      HRT  22
C                         CASE C, P. 344 OF HANTUSH.                    HRT  23
```

TABLE 11.1.—*Listing of program to compute the convolution integral for a leaky aquifer*—Continued

```
C                  IQ=4, Q(T) IS A FIFTH-DEGREE POLYNOMIAL.
C                  IQ=5, Q(T) IS A PIECEWISE LINEAR FUNCTION OF
C                       TIME (EIGHT SEGMENTS).
C             QR = REFERENCE DISCHARGE, ZERO OR BLANK FOR PROJECTION.
C        1 OR 4 CARDS, DEPENDING ON IQ.
C             IF IQ=1,2,3 = 1 CARD = FORMAT(3E10.3)
C                  QST = EVENTUAL CONSTANT DISCHARGE.
C                  DELTA = RATE PARAMETER.
C                  TSTAR = TIME PARAMETER.
C             IF IQ=4 = 1 CARD = FORMAT(6E10.3)
C                  AQ(6) = 6 VALUES = THE POLYNOMIAL COEFFICIENTS
C                       WITH A0 FIRST AND A5 LAST.
C             IF IQ=5 = 4 CARDS = FORMAT(6E10.3)
C                  TI(I),AI(I),BI(I),TI(I+1),AI(I+1),BI(I+1),I=1,3,5,7
C                  PARAMETERS OF THE PIECEWISE LINEAR FUNCTION
C                  (8 SEGMENTS), CODED 2 SEGMENTS PER CARD, FIRST
C                  AND SECOND SEGMENTS ON FIRST CARD, THEN SEQUENTIALLY
C                  ON SUCCEEDING CARDS. EACH SEGMENT HAS THREE
C                  PARAMETERS WHICH ARE IN CODING ORDER
C                       TI = ENDING TIME OF THE SEGMENT.
C                       AI = DISCHARGE AT BEGINNING OF SEGMENT.
C                       BI = RATE OF CHANGE IN DISCHARGE DURING SEG.
C                  THE DISCHARGE FUNCTION IN EACH SEGMENT HAS THE
C                  FORM Q(T) = AI(I)+BI(I)*(T-TI(I-1)). IF LESS THAN 8
C                  SEGMENTS ARE NEEDED, BLANKS CAN BE CODED FOR
C                  SUCCEEDING SEGMENTS.
C        2 OR MORE CARDS = FORMAT(4E10.3)
C             R = RADIAL DISTANCE FROM PUMPED WELL, BLANK OR ZERO
C                  SIGNALS PROGRAM AS END TO GROUP OF DATA.
C             S = STORAGE COEFFICIENT
C             T = TRANSMISSIVITY
C             PM = (P'/M') = HYD. COND. OF CONFINING BED DIVIDED
C                  BY THICKNESS OF CONFINING BED.
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        CONVOL,Q = MUST BE INCLUDED IN DECK.
C
C***********************************************************************
      DIMENSION       D(12),IEX(12),X(6),H(12,6),QS(12,6),CP(12),CT(12)
      DIMENSION H1(12),H2(12),Q1(12),Q2(12)
      DIMENSION H3(12),H4(12),Q3(12),Q4(12)
      COMMON AQ(6),TI(9),AI(9),BI(9),QST,DELTA,TSTAR
      DATA CP/12*'  T*'/,CT/12*'1/U*'/,D/12*'10**'/
      DATA H1/12*'  S('/,H2/12*'R,T)'/,Q1/12*'    '/,Q2/12*'Q(T)'/
      DATA H3/12*'   S'/,H4/12*'Q(T)'/,Q3/12*' Q(T'/,Q4/12*')/QR'/
      DATA X/1.,1.5,2.,3.,5.,7./
      TI(1)=0.
      N=500
    1 READ (5,18,END=17) TBEGIN,TEND,IQ,QR
      IF (IQ.LT.4) READ (5,19) QST,DELTA,TSTAR
      IF (IQ.EQ.4) READ (5,19) AQ
      IF (IQ.EQ.5) READ (5,19) (TI(I),AI(I),BI(I),I=2,9)
      WRITE (6,24)
    2 READ (5,19) R,S,T,PM
      IF (R.EQ.0.) GO TO 1
      A=R*R*S/(4.*T)
      B=PM/S
      Y=ALOG10(TBEGIN)
```

TABLE 11.1.—*Listing of program to compute the convolution integral for a leaky aquifer*—Continued

```
      IF (Y) 3,5,4
    3 Y=Y-.001
      GO TO 5
    4 Y=Y+.001
    5 IBEGIN=Y
      Y=ALOG10(TEND)
      IF (Y) 6,8,7
    6 Y=Y-.001
      GO TO 8
    7 Y=Y+.001
    8 IEND=Y
      M=IEND-IBEGIN+1
      IF (M.GT.12) M=12
      DO 10 I=1,M
      IEX(I)=IBEGIN+I-1
      Y=10.**(IBEGIN+I-1)
      DO 10 J=1,6
      TIME=X(J)*Y
      IF (QR.GT.0.) TIME=A*TIME
      CALL CONVOL(TIME,A,B,N,IQ,SUM)
      IF (QR.GT.0.) GO TO 9
      H(I,J)=SUM/(12.5664*T)
      QS(I,J)=Q(TIME,IQ)
      GO TO 10
    9 H(I,J)=SUM/QR
      QS(I,J)=Q(TIME,IQ)/QR
   10 CONTINUE
      K=M
      IF (M.GT.6) K=6
      IF (QR.GT.0.) GO TO 11
      WRITE (6,20) A,B,(CP(I),D(I),IEX(I),I=1,K)
      WRITE (6,21) (H1(I),H2(I),Q1(I),Q2(I),I=1,K)
      GO TO 12
   11 WRITE (6,25) A,B,QR,(CT(I),D(I),IEX(I),I=1,K)
      WRITE (6,21) (H3(I),H4(I),Q3(I),Q4(I),I=1,K)
   12 DO 13 J=1,6
      WRITE (6,22) X(J),(H(I,J),QS(I,J),I=1,K)
   13 CONTINUE
      IF (M.LE.6) GO TO 2
      K1=K+1
      IF (QR.GT.0.) GO TO 14
      WRITE (6,23) (CP(I),D(I),IEX(I),I=K1,M)
      WRITE (6,21) (H1(I),H2(I),Q1(I),Q2(I),I=K1,M)
      GO TO 15
   14 WRITE (6,26) (CT(I),D(I),IEX(I),I=K1,M)
      WRITE (6,21) (H3(I),H4(I),Q3(I),Q4(I),I=K1,M)
   15 DO 16 J=1,6
      WRITE (6,22) X(J),(H(I,J),QS(I,J),I=K1,M)
   16 CONTINUE
      GO TO 2
   17 STOP
C
   18 FORMAT (2E10.5,4X,I1,5X,E10.5)
   19 FORMAT (6E10.3)
   20 FORMAT ('0','R**2*S/(4*TRANS)=',1PE10.3,', K''/(S*B'')=',E10.3/'0'
     1,2X,'T',5X,6(2A4,I2,9X))
   21 FORMAT (' ',4X,6(2A4,2X,2A4,1X))
```

TABLE 11.1.—*Listing of program to compute the convolution integral for a leaky aquifer*—Continued

```
   22 FORMAT (' ',F4,1,6(0PF8,3,1PE11,3))
   23 FORMAT ('0',2X,'T',5X,6(2A4,I2,9X))
   24 FORMAT (1H1)
   25 FORMAT ('0','R**2*S/(4*TRANS)=',1PE10,3,', K''/(S*B'')=',E10,3,',
     1QR=',E10,3/'0',1X,'1/U',4X,6(2A4,I2,9X))
   26 FORMAT ('0',1X,'1/U',4X,6(2A4,I2,9X))
      END

      SUBROUTINE CONVOL(TIME,A,B,N,IQ,SUM)
C*********************************************************************
C
C     PURPOSE
C        COMPUTES VALUES OF THE CONVOLUTION INTEGRAL FOR LEAKY
C        AQUIFERS, THE INTEGRAL IS, FROM 0 TO T, OF
C        Q(T-T')/T'*EXP(-A/T'-B*T')*DT',
C     DESCRIPTION OF PARAMETERS
C        A,B,SUM ARE REAL; N,IQ ARE INTEGER,
C        A - R**2*S/(4*T) - RADIAL DISTANCE SQUARED * STORAGE
C               COEFFICIENT / 4 * TRANSMISSIVITY,
C        B - P'/(S*M') - HYD, COND, OF CONFINING BED DIVIDED BY
C               AQUIFER STORAGE COEFFICIENT * THICKNESS OF CONF, BED,
C        N - NUMBER OF INCREMENTS FOR EACH INTERVAL OF THE SUM,
C        IQ - INDICATES FORM OF DISCHARGE FUNCTION,
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        Q
C     METHOD
C        APPROXIMATES INTEGRAL BY SUMMING THE TRAPEZOIDAL RULE APPLIED
C        TO A SEQUENCE OF SEGMENTS, LOWER LIMIT OF FIRST SEGMENT IS
C        PICKED AT POINT WHERE EXPONENT > -100 ,
C        IF SUCH A POINT DOES NOT EXIST (A*B > 2500) A FUNCTION VALUE
C        OF 0 IS RETURNED, UPPER LIMIT = 10 * LOWER LIMIT FOR EACH
C        SEGMENT, USES INCREMENT OF DELTA T' = (U-L)/N WHERE N IS THE
C        NUMBER OF INCREMENTS IN THE CALL, CEASES SUMMATION WHEN
C        EXPONENT < -101 ,
C
C*********************************************************************
      REAL*8 DSUM
      REAL*4 NEWT,NEWTP,NEWX,NEWF
      DSUM=0,D+0
      IS=0
C     INITIAL T' COMPUTED FROM A,B
      AB=A*B
      IF (AB,GE,2500,) GO TO 7
      IF (B,GT,0,) GO TO 2
    1 OLDT=,01*A
      GO TO 3
    2 OLDT=(1,-SQRT(1,-AB/2500,))*50,/B
      IF (OLDT,EQ,0,) GO TO 1
C     INITIAL T-T'
    3 OLDTP=TIME-OLDT
      OLDX=-A/OLDT-B*OLDT
      OLDF=Q(OLDTP,IQ)*EXP(OLDX)/OLDT
C     END OF SUMMATION SEGMENT IS 10 TIMES THE BEGINNING
    4 ENDT=10,*OLDT
      IF (ENDT,LT,TIME) GO TO 5
      IF (OLDT,GE,TIME) GO TO 7
      IS=1
      ENDT=TIME
```

TABLE 11.1.—*Listing of program to compute the convolution integral for a leaky aquifer*—Continued

```
C     DELTA T' IS COMPUTED FROM LENGTH AND NUMBER OF INCREMENTS
    5 DELT=(ENDT-OLDT)/N
      DO 6 I=1,N
C     T' IS INCREMENTED BY DELTA T'
      NEWT=OLDT+DELT
      NEWX=-A/NEWT-B*NEWT
C     TERMINATES SUMMATION WHEN EXP(-A/T'-B*T') < 1.37E-44
      IF (NEWX.LT.-101.) GO TO 7
      NEWTP=TIME-NEWT
      NEWF=Q(NEWTP,IQ)*EXP(NEWX)/NEWT
      DSUM=DSUM+(NEWF+OLDF)*DELT
      OLDT=NEWT
      OLDF=NEWF
    6 CONTINUE
      IF (IS.GT.0) GO TO 7
C     IF T' < T, BEGINS A NEW SEGMENT
      GO TO 4
    7 SUM=DSUM/2.D+0
      RETURN
      END
```

```
      FUNCTION Q(TIME,IQ)
C***********************************************************************
C
C     PURPOSE
C        COMPUTES THE DISCHARGE FUNCTION, Q(T)
C     DESCRIPTION OF PARAMETERS
C        TIME - REAL - ELAPSED TIME SINCE BEGINNING OF DISCHARGE.
C        IQ - INTEGER - INDICATES FORM OF DISCHARGE FUNCTION.
C             IQ=1,2,3, CASES A,B,C, RESPECTIVELY, OF HANTUSH,M.S.,
C                  1964, HYDRAULICS OF WELLS IN CHOW, VEN TE, ED.,
C                  ADVANCES IN HYDROSCIENCE, VOL. 1: ACADEMIC PRESS,
C                  NEW YORK, P. 343,344.
C             IQ=4, DISCHARGE IS A FIFTH DEGREE POLYNOMIAL OF TIME.
C             IQ=5, DISCHARGE IS A PIECEWISE LINEAR FUNCTION OF UP TO
C                  8 SEGMENTS.
C     METHOD
C        FORTRAN EVALUATION OF FUNCTIONS.
C
C***********************************************************************
      COMMON AQ(6),TI(9),AI(9),BI(9),QST,DELTA,TSTAR
      GO TO (1,2,3,4,5), IQ
    1 Q=QST*(1.+DELTA*EXP(-TIME/TSTAR))
      RETURN
    2 Q=QST*(1.+DELTA/(1.+TIME/TSTAR))
      RETURN
    3 Q=QST*(1.+DELTA/SQRT(1.+TIME/TSTAR))
      RETURN
    4 Q=AQ(1)+TIME*(AQ(2)+TIME*(AQ(3)+TIME*(AQ(4)+TIME*(AQ(5)+TIME*AQ(6)
     1))))
      RETURN
    5 DO 6 I=2,9
      IF (TIME.LE.TI(I)) GO TO 7
    6 CONTINUE
      I=9
    7 Q=AI(I)+BI(I)*(TIME-TI(I-1))
      RETURN
      END
```