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Techniques of Water-Resources Investigations
of the United States Geological Survey

Chapter B2

INTRODUCTION TO GROUND-WATER HYDRAULICS

A Programed Text for Self-Instruction

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Book 3

APPLICATIONS OF HYDRAULICS

Part VI. Nonequilibrium Flow to a Well

Introduction

In Part V we developed the equation

$$\frac{\partial^2 h}{\partial x^2}=\frac{S}{T}\frac{\partial h}{\partial t}$$

for one-dimensional nonequilibrium flow in a homogeneous and isotropic confined aquifer. We indicated, in addition, that extension to two-dimensional flow would yield the equation

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

In Part VI we consider a problem involving flow away from (or toward) a well in such an aquifer. As in the steady-state problem of flow to a well, which we considered in Part III, we will find it convenient here to use polar coordinates. The two-dimensional differential equation

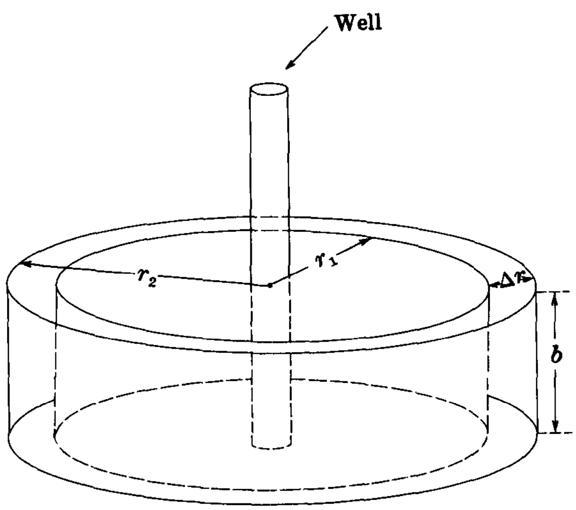
$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}$$

can be transformed readily into polar coordinates by using standard methods. However, it is both easy and instructive to derive the equation again from hydraulic principles in the form in which we are going to use it. After we have developed the differential equation in this way, we will consider one of its solutions, corresponding to an instantaneous disturbance to the aquifer. In the terminology of systems analysis, this solution will give the "impulse response" of the well-aquifer system. In considering this solution, we will first show by differentiation that it satisfies the given differential equation; we will then develop the boundary conditions applicable to the problem and show that the solution satisfies these conditions. Following the programed section of Part VI, a discussion in text format has been added showing how the "impulse response" solution may be used to synthesize solutions corresponding to more complex disturbances to the aquifer. In particular, solutions are synthesized for the case of repeated withdrawal, or bailing, of a well and for the case of continuous pumping of a well. The latter solution, for the particular case in which the pumping rate is constant, is the Theis equation, which is commonly used in aquifer test analysis.

1 +

The figure shows a well penetrating a confined aquifer. A cylindrical shell or prism, coaxial with the well and extending through the full thickness, b, of the aquifer has been outlined in the diagram. The radial width of this cylindrical element is designated Δr; the inner surface of the element is at a radius r_1 from the axis of the well, which is taken as the origin of the polar coordinate system; and the outer surface of the element is at a radius r_2 from this axis. We assume all flow to be in the radial direction, so that

1 + —Con.



we need not consider variation in the vertical or angular directions. We further assume that we are dealing with injection of water into the aquifer through the well, so that flow is outward, away from the well, in the positive r direction. The hydraulic conductivity of the aquifer is denoted K, the transmissivity T, and the storage coefficient S.

QUESTION

If $(\partial h/\partial r)_1$ represents the hydraulic gradient at the inner face of the cylindrical element, which of the following expressions will be obtained for the flow through this face, by an application of Darcy's law?

Turn to Section:

$$Q_1 = -K\pi r_1^2\left(\frac{\partial h}{\partial r}\right)_1$$ 34

$$Q_1 = -K2\pi r_1 b\left(\frac{\partial h}{\partial r}\right)_1$$ 15

$$Q_1 = \frac{-Kb\left(\frac{\partial h}{\partial r}\right)_1}{2\pi r_1}$$ 36

2 +

Your answer in Section 27,

$$\frac{\partial h}{\partial r} = \frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)},$$

is not correct.

You are correct in your intention to multiply the derivative of $e^{-(Sr^2/4Tt)}$ by the "constant" coefficient $V/(4\pi Tt)$ to obtain the derivative of the product

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)},$$

with respect to r. However, your differentiation of $e^{-(Sr^2/4Tt)}$ is not correct. The derivative of e raised to some power is not simply e raised to the same power, as you have written, but the product of e raised to that power times the derivative of the exponent. That is,

$$\frac{de^u}{dr} = e^u\frac{du}{dr}.$$

Thus, in this case, we must obtain the derivative of the exponent, $-(Sr^2/4Tt)$, and multiply $e^{-(Sr^2/4Tt)}$ by this derivative to obtain the derivative of $e^{-(Sr^2/4Tt)}$ with respect to r.

Return to Section 27 and choose another answer.

3 +

Your answer in Section 35,

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\left(\frac{Sr^2}{4Tt^2}\right),$$

is not correct. In your answer, the term $e^{-(Sr^2/4Tt)}$ is differentiated correctly with respect to time. However, your answer gives only the derivative of this factor times the first factor itself, $V/(4\pi Tt)$. According to the rule for differentiation of a product, we must add to this the second factor, $e^{-(Sr^2/4Tt)}$, times the derivative of the first factor. The first factor, $V/(4\pi Tt)$ was treated as a constant coefficient when we were differentiating with respect to r, since it does not contain r. It does, however, contain t and cannot be treated as a constant when we are differentiating with respect to t. Its derivative with respect to t is given in the discussion of Section 35.

Return to Section 35 and choose another answer.

4 +

Your answer in Section 27,

$$\frac{\partial h}{\partial r}=e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right),$$

is not correct.

When an expression is multiplied by a constant coefficient, the derivative of the product is simply the constant coefficient times the derivative of the expression. For example, the derivative of the expression x^2, with respect to x, is $2x$; but if x^2 is multiplied by the constant coefficient c, the derivative of the product, cx^2, is $c\cdot 2x$.

In the question of Section 27, the term $e^{-(Sr^2/4Tt)}$ is actually the expression in which we must differentiate with respect to r. The term $V/(4\pi Tt)$, represents a constant coefficient—constant with respect to this differentiation, because it does not contain r. Thus whatever we obtain as the derivative of $e^{-(Sr^2/4Tt)}$ must be multiplied by this coefficient, $V/(4\pi Tt)$, to obtain the derivative of the product

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}.$$

Your differentiation of $e^{-(Sr^2/4Tt)}$ is correct, but your answer does not contain the factor $V/(4\pi Tt)$ and thus cannot be correct.

Return to Section 27 and choose another answer.

5 +

Your answer in Section 27,

$$\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right),$$

is correct.

We now wish to differentiate this expression for $\partial h/\partial r$, in order to obtain $\partial^2h/\partial r^2$. To do this, we treat the expression as the product of two factors. The first is the function we just differentiated,

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)};$$

the second is

$$\left(\frac{-2Sr}{4Tt}\right).$$

Once again we are differentiating with respect to r, so that t is treated as a constant.

5 + —Con.

QUESTION

If we follow the rule for differentiation of a product (first factor times derivative of second, plus second factor times derivative of first), which of the following results do we obtain for $\partial^2 h/\partial r^2$?

Turn to Section:

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\left\{e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right)\right\}$$ 35

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right)$$ 23

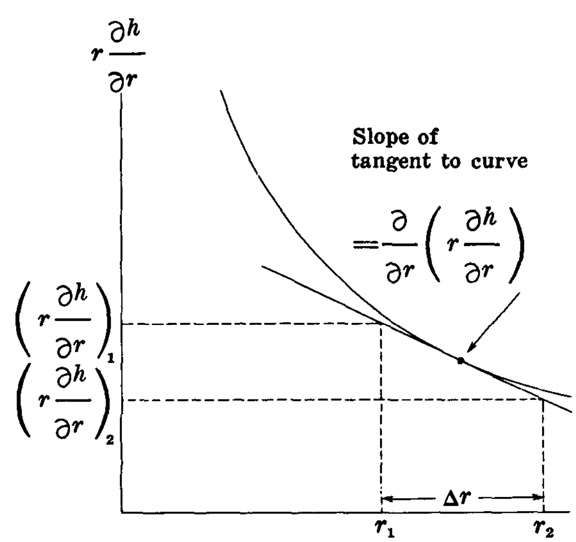
$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\left\{e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot e^{-(Sr^2/4Tt)}\right\}$$ 9

6 +

Your answer in Section 18 is not correct. The answer which you chose states that head becomes infinite as radial distance becomes small. The behavior which we are trying to describe is that in which head dies out, or approaches zero, as radial distance becomes very large.

Return to Section 18 and choose another answer.

7 +



Your answer in Section 15,

$$Q_1-Q_2=2\pi T\left\{\left(r\frac{\partial h}{\partial r}\right)_2-\left(r\frac{\partial h}{\partial r}\right)_1\right\},$$

is correct. The term

$$\left\{\left(r\frac{\partial h}{\partial r}\right)_2-\left(r\frac{\partial h}{\partial r}\right)_1\right\}$$

actually represents the change in the variable $r(\partial h/\partial r)$ between the radial limits, r_1 and r_2, of our element. If we imagine a plot of $r(\partial h/\partial r)$ versus r, as in the figure, we can readily see that this change will be given approximately by the slope of the plot times

(*continued on next page*)

7 + —Con.

the radial increment, Δr. That is, approximately

$$\left(r\frac{\partial h}{\partial r}\right)_2 - \left(r\frac{\partial h}{\partial r}\right)_1 = \frac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r}\cdot\Delta r$$

where the derivative

$$\frac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r}$$

represents the slope of our plot, at an appropriate point within the element. This slope, or derivative, is negative in our illustration, so that

$$\left(r\frac{\partial h}{\partial r}\right)_1 > \left(r\frac{\partial h}{\partial r}\right)_2.$$

The approximation inherent in the above equation becomes progressively more accurate as Δr decreases in size.

QUESTION

Recalling that the rule for differentiation of a product is "first factor times derivative of second plus second factor times derivative of first," which of the following equations gives the derivative of $r(\partial h/\partial r)$ with respect to r?

	Turn to Section:
$\dfrac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r} = r\dfrac{\partial\left(\dfrac{\partial h}{\partial r}\right)}{\partial r} + \dfrac{\partial h}{\partial r}r$	26
$\dfrac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r} = r\dfrac{\partial_2 h}{\partial r^2} + \dfrac{\partial h}{\partial r}$	28
$\dfrac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r} = 2r\dfrac{\partial^2 h}{\partial r^2}$	8

8 +

Your answer in Section 7,

$$\frac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r} = 2r\frac{\partial^2 h}{\partial r^2},$$

is not correct. We are required to take the derivative of the product $r(\partial h/\partial r)$. The rule for differentiation of a product is easy to remember: first factor times derivative of second, plus second factor times derivative of first; that is

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

A derivation of this formula can be found in any standard text of calculus. Our first factor is r, and our second factor is $\partial h/\partial r$. Thus we must form the expression: r times the derivative of $\partial h/\partial r$ with respect to r, plus $\partial h/\partial r$ times the derivative of r with respect to r.

Return to Section 7 and choose another answer.

9 +

Your answer in Section 5,

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\left\{e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot e^{-(Sr^2/4Tt)}\right\},$$

is not correct. If we remove the braces and separate your answer into two terms, we have

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}.$$

The first term, according to the rule for differentiation of a product, is correct, since it represents the first factor,

$$\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}$$

multiplied by the derivative of the second (with respect to r), which is simply

$$\frac{-2S}{4Tt}.$$

The second term of your answer, however, is not correct.

$$\frac{-2Sr}{4Tt}$$

is the second factor of the product we wish to differentiate but

$$\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}$$

does not represent the derivative of the first factor. This first factor is itself

$$\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}$$

and its derivative with respect to r was obtained in answer to the question of Section 27.

Return to Section 5 and choose another answer.

10 +

Your answer in Section 21 is not correct. We established in the discussion of Section 21 that the rise in head within the well at $t=0$, due to injection of the volume V, would be given by V/A_w, where A_w is the cross-sectional area of the well bore. If the well radius approaches zero, A_w must approach zero. The smaller A_w becomes, the larger the quotient V/A_w must become; for example, 1/0.001 is certainly much greater than 1/1. Your answer, that the head change is zero, could only be true if the area of the well were immeasurably large, so that the addition of a finite volume of water would produce no measurable effect.

Return to Section 21 and choose another answer.

11 +

Your answer in Section 33 is not correct. The integration in the equation

$$V=\int_{r=0}^{r=\infty} S\cdot h_{r,t}\cdot 2\pi r dr$$

cannot be carried out until we substitute some clearly defined function of r for the term $h_{r,t}$. Until this is done, we do not even know what function we are trying to integrate. But even if the integration could be carried out and the result were found to be

$$\frac{V}{4\pi Tt} e^{-(r^2S/4Tt)}$$

then we would be left with the result

$$V=\frac{V}{4\pi Tt} e^{-(r^2S/4Tt)}$$

which clearly can never be satisfied except perhaps at isolated values of r and t.

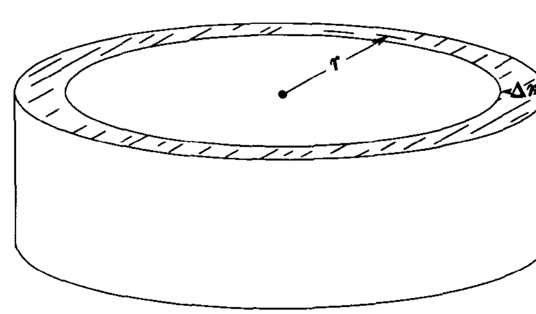
Return to Section 33 and choose another answer.

12 +

Your answer in Section 28,

$$\frac{dV}{dt}=S\pi r^2\frac{\partial h}{\partial t},$$

is not correct. The storage equation states that the rate of accumulation in storage is equal to the product of storage coefficient, rate of change of head with time, and base area of the element (prism) of aquifer under consideration. Your answer contains the storage coefficient, S, and the time rate of change, $\partial h/\partial t$. However, the base area of the prism which we are considering is not given by πr^2.



This term gives the area of a circle extending from the origin to the radius r; our prism is actually a cylindrical shell, extending from the radius r_1 to the radius r_2. Its base area is the area of the shaded region in the figure. This region has a radial width of Δr and a mean perimeter of $2\pi r$.

Return to Section 28 and choose another answer.

13 +

Your answer in Section 33 is correct. Our proposed solution, giving h as a function of r and t is

$$h_{r,t}=\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}.$$

To test this solution for conformity with the required condition we substitute

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

for $h_{r,t}$ in the equation

$$V=\int_{r=0}^{r=\infty}S\cdot h_{r,t}\cdot 2\pi rdr$$

and evaluate the integral to see whether the equation is satisfied. The substitution gives

$$V=\int_{r=0}^{r=\infty}S\cdot\frac{V}{4\pi Tt}\cdot e^{-(r^2S/4Tt)}\cdot 2\pi rdr.$$

Constant terms may be taken outside the integral; in this case, we are integrating with respect to r, so t may be treated as a constant and taken outside the integral as well. We leave the factor 2 under the integral for the moment and take the remaining constants outside to give

$$V=\frac{SV}{4Tt}\int_{r=0}^{r=\infty}e^{-(r^2S/4Tt)}\cdot 2rdr.$$

To evaluate the integral in this form, we make use of a simple algebraic substitution. Let

$$z=r^2;$$

then

$$dz=2rdr;$$

and let

$$a=\frac{S}{4Tt}.$$

Substituting these terms in the above equation, we obtain:

$$V=aV\int_{z=0}^{z=\infty}e^{-az}dz.$$

The indefinite integral of e^{-az} is simply

$$-\frac{1}{a}e^{-az};$$

that is,

$$\int e^{-az}dz=-\frac{1}{a}e^{-az}+c$$

where c is a constant of integration. The infinite upper limit in our problem is handled by the standard method; the steps are as follows

$$\int_{z=0}^{z=\infty}e^{-az}dz=\lim_{b\to\infty}\int_0^b e^{-az}dz$$

$$=\lim_{b\to\infty}\left\{-\frac{1}{a}e^{-az}\Big|_{z=0}^{z=b}\right\}=\lim_{b\to\infty}\left\{-\frac{1}{a}\cdot\frac{1}{e^{ab}}-\left(-\frac{1}{a}\cdot\frac{1}{e^0}\right)\right\}$$

$$=-\frac{1}{a}\left(\lim_{b\to\infty}\left\{\frac{1}{e^{ab}}\right\}\right)+\frac{1}{a}$$

but

$$\lim_{b\to\infty}\left\{\frac{1}{e^{ab}}\right\}=0,$$

so that

$$\int_{z=0}^{z=\infty}e^{-az}dz=\frac{1}{a}.$$

Therefore

$$aV\int_{z=0}^{z=\infty}e^{-az}dz=aV\cdot\frac{1}{a}=V.$$

This verifies that our function

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

(*continued on next page*)

13 + —Con.

actually satisfies the required condition—that is, that when we substitute this term for $h_{r,t}$ in the expression

$$\int_{r=0}^{r=\infty} S \cdot h_{r,t} \cdot 2\pi r dr$$

and perform the integration, the result is actually equal to V, the volume of injected water, as required by the condition.

We have shown, then, that the expression

$$h=\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

satisfies the differential equation for radial flow in an aquifer and satisfies as well the boundary conditions associated with the instantaneous injection of a volume of water through a well at the origin, at $t=0$. It is, therefore, the particular solution required for this problem. It is an important solution for two reasons. First, it describes approximately what happens when a charge of water is suddenly added to a well in the standard "slug test" (Ferris and Knowles, 1963) and provides a means of estimating transmissivity through such a test.[1] Second, and more importantly, it gives the "impulse response" of the well-aquifer system—the solution corresponding to an instantaneous disturbance. Solutions for more complicated forms of disturbance, such as repeated injections or withdrawals. or continuous withdrawal, can be synthesized from this elementary solution. Following Section 37, a discussion is given in text format outlining the manner in which solutions corresponding to repeated bailing and continuous pumping of a well may be built up from the impulse response solution.

This concludes the programed instruction of Part VI. You may proceed to the text-format discussion following Section 37. Readers who prefer may proceed to Part VII.

[1] A subsequent publication (Cooper, Bredehoeft, and Papadopulos, 1967) has provided a more accurate description of the actual effect of adding a charge of water to a well, by considering the inertia of the column of water in the well. This factor was neglected in the original analysis.

14 +

Your answer in Section 33 is not correct. The condition to be satisfied was

$$V=\int_{r=0}^{r=\infty} S \cdot h_{r,t} \cdot 2\pi r dr.$$

A solution to our differential equation is by definition an expression giving the head, h, at any radius, r, and time, t, in a form that satisfies the differential equation. Here, the idea is to test such a solution to see if it *also* satisfies the condition phrased in the above equation. The solution actually represents the head, $h_{r,t}$; if we substitute it for the quantity $2\pi r$, as your answer suggests, there will be two terms, $h_{r,t}$ and our solution, both representing head in the resulting equation. Moreover if the result of the integration were $2\pi S$ we would be left with the result $V=2\pi S$, which does not satisfy the required condition.

Return to Section 33 and choose another answer.

15 +

Your answer in Section 1,

$$Q_1=-K2\pi r_1 b\left(\frac{\partial h}{\partial r}\right)_1,$$

is correct. The terms 2π, K, and b are all constants; we will denote the product Kb by T, as before. The variable terms, r and $\partial h/\partial r$, may be combined and treated as a single variable, $r(\partial h/\partial r)$. The value of this variable at the inner face of the cylindrical element will be designated $(r\partial h/\partial r)_1$. Using these notations, our expression for inflow

15 + —Con.

through the inner face of the cylindrical element is now

$$Q_1 = -2\pi T\left(r\frac{\partial h}{\partial r}\right)_1 .$$

QUESTION

Suppose we continue to treat the product $r(\partial h/\partial r)$ as a single variable, and let $(r\partial h/\partial r)_2$ denote the value of this variable at the outer face of the cylindrical element. The expression for the outflow, Q_2, through the outer cylindrical surface can then be written in terms of $(r\partial h/\partial r)_2$, in a form similar to that for the inflow. Which of the following equations would we then obtain for the inflow minus outflow, $Q_1 - Q_2$, for our cylindrical element?

	Turn to Section:
$Q_1 - Q_2 = 2\pi T\left\{\left(r\frac{\partial h}{\partial r}\right)_2 - \left(r\frac{\partial h}{\partial r}\right)_1\right\}$	7
$Q_1 - Q_2 = 2\pi T\left(r\frac{\partial h}{\partial r}\right)_1 \cdot \left(r\frac{\partial h}{\partial r}\right)_2$	30
$Q_1 - Q_2 = 2\pi T\left\{\left(\frac{\partial h}{\partial r}\right)_2 - \left(\frac{\partial h}{\partial r}\right)_1\right\}$	25

16 +

Your answer in Section 28,

$$\frac{dV}{dt} = \frac{S\frac{\partial h}{\partial t}}{2\pi r\Delta r},$$

is not correct. The storage equation tells us that rate of accumulation in storage should equal the *product* of storage coefficient, rate of change of head with time, and base area of the element (prism) of aquifer with which we are dealing. Our element, or prism, of aquifer is a cylindrical shell extending from the radius r_1 to the radius r_2. Its base area is given by the term $2\pi r\Delta r$. However, in your answer this area term is divided into the term $S(\partial h/\partial t)$.

Return to Section 28 and choose another answer.

17 +

Your answer in Section 20,

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-2S}{4Tt} + \frac{2S^2r^2}{16T^2t^2}\right\},$$

is not correct. The mistake in this answer results from an algebraic error in simplifying the second term of the expression for $\partial^2 h/\partial r^2$. The product

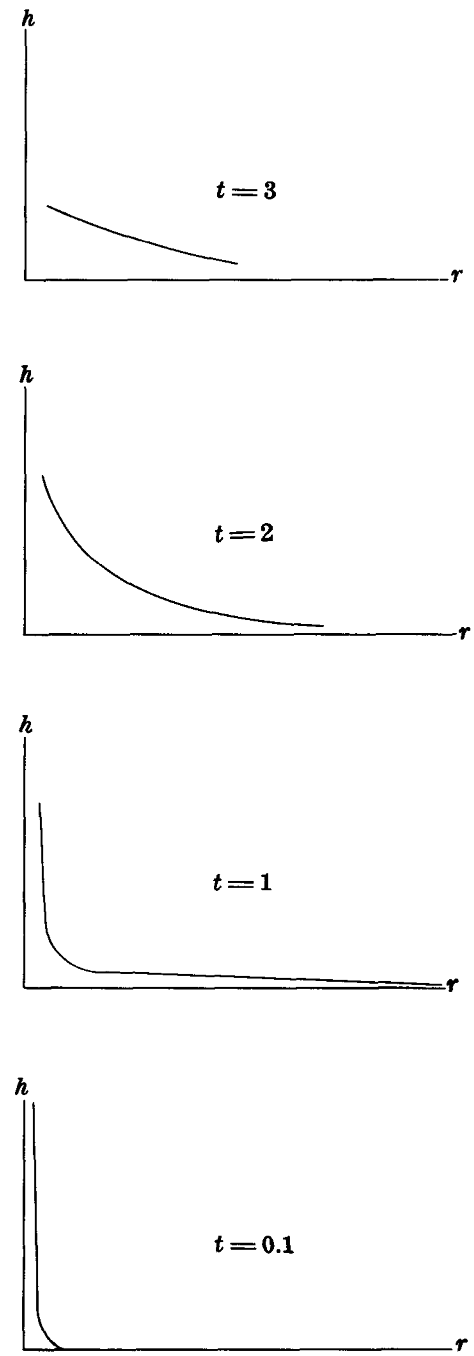
$$\left(\frac{-2Sr}{4Tt}\right)\cdot\left(\frac{-2Sr}{4Tt}\right)$$

is not equal to

$$\frac{2S^2r^2}{16T^2t^2}.$$

Return to Section 20 and choose another answer.

18 +



Your answer in Section 21 is correct; head is immeasurably great, or infinite, at the well at $t=0$. Taking this result together with our requirement that head must be zero elsewhere in the aquifer at $t=0$, we may phrase the boundary condition for $t=0$ as follows

$$h \to \infty, \text{ for } r=0 \text{ and } t=0$$
$$h=0, \text{ for } r>0 \text{ and } t=0.$$

We now test our solution to see if it satisfies this requirement. Probably the easiest way to do this is to expand the term $e^{-(Sr^2/4Tt)}$ in a Maclaurin series. The theory of this type of series expansion is treated in standard texts of calculus; the result, as applied to our exponential function, has the form

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ***$$

or for a negative exponent,

$$e^{-x} = \frac{1}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ***}.$$

In our case, x is the term $r^2S/4Tt$, and

$$e^{-(r^2S/4Tt)} = \frac{1}{1 + \left(\frac{r^2S}{4Tt}\right) + \frac{\left(\frac{r^2S}{4Tt}\right)^2}{2!} + \frac{\left(\frac{r^2S}{4Tt}\right)^3}{3!} + ***}$$

so that

$$\frac{V}{4\pi Tt} e^{-(r^2S/4Tt)} =$$

$$\frac{V}{4\pi Tt + r^2S\pi + \frac{r^4S^2\pi}{4Tt \cdot 2!} + \frac{r^6S^3\pi}{16T^2t^2 \cdot 3!} + ***}.$$

Now as t approaches zero, the first term in the denominator approaches zero; the second remains constant, and the third and all higher terms become infinite, provided r does not also approach zero. If any term in the

18 + —Con.

denominator is infinite, the fraction as a whole becomes zero. Thus the expression

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

is zero for $t=0$ and $r\neq 0$, and satisfies the first part of our condition.

If r and t are both allowed to approach zero, the first two terms in the denominator of our fraction will be zero. The third will behave in the same manner as the fraction cx^4/kx behaves as x approaches zero, since r and t are both approaching zero in the same way. The limit of cx^4/kx as x approaches zero is 0, since

$$\frac{cx^4}{kx}=\frac{c}{k}x^3.$$

Therefore the third term in the denominator must also approach the limit zero as r and t approach zero. By a similar analysis it can be shown that the limit of every succeeding term in the denominator is zero as r and t approach zero. Thus the entire denominator is zero, and the fraction as a whole is infinite, so that the term

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

is infinite when r and t are both zero, satisfy- the second part of our condition.

Another and very instructive way to investigate the behavior of the function

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

is to construct plots of this function versus r, for decreasing values of time. The figures show the form that such a series of plots will take. It may be noted that as time approaches zero the function approaches the shape of a sharp "spike," or impulse, at $r=0$. The shape of these curves suggests a head distribution which we might sketch intutively, if we were asked to describe the response of an aquifer to the injection of a small volume of water. It is suggested that the reader construct a few of these plots, in order to acquire a feeling for the behavior of the function.

QUESTION

The aquifer is assumed to be infinite in extent, and the volume of water injected is assumed to be small. We would therefore expect the effects of the injection to die out at great radial distances from the well. Which of the following expressions is a mathematical formulation of this behavior and could be used as a boundary condition for our problem?

	Turn to Section:
$h\to 0$ as $r\to\infty$	33
$h\to\infty$ as $t\to\infty$	29
$h\to\infty$ as $r\to 0$	6

19 +

Your answer in Section 21 is not correct. We established in the discussion of Section 21 that the rise in water level in the well at $t=0$ should be given by the expression $h=V/A_w$, where A_w is the cross-sectional area of the well bore and V is the volume of water injected. In order for h to have the instantaneous value of 1 foot, V, in cubic feet, would have to be numerically equal to A_w, in square feet. However, we are assuming the well to have an infinitesimally small radius, so that A_w, its cross-sectional area, approaches zero. If smaller and smaller values are assigned to the denominator, A_w, while the numerator, V, is held constant, the fraction V/A_w must take on larger and larger values.

Return to Section 21 and choose another answer.

20+

Your answer in Section 35,

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left(\frac{Sr^2}{4Tt^2}\right)+e^{-(Sr^2/4Tt)}\left(\frac{-V}{4\pi Tt^2}\right),$$

is correct. If the term

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}$$

is factored from this expression we have

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{Sr^2}{4Tt^2}-\frac{1}{t}\right\}$$

and if we multiply this equation by S/T, we obtain

$$\frac{S}{T}\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{S^2r^2}{4T^2t^2}-\frac{S}{Tt}\right\}.$$

Our expression for $\partial h/\partial r$, obtained in answer to the question of Section 27 was

$$\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left(\frac{-2Sr}{4Tt}\right).$$

The term $(1/r)(\partial h/\partial r)$ is therefore given by

$$\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right).$$

In answering the question of Section 5, we saw that the expression for $\partial^2h/\partial r^2$ was

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\left\{e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)e^{-(Sr^2/4Tt)}\left(\frac{-2Sr}{4Tt}\right)\right\}$$

20 + —Con.

QUESTION

Which of the following expressions is obtained for

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r},$$

by combining the two expressions given above and factoring out the term

$$\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\ ?$$

	Turn to Section:
$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-S}{Tt}+\frac{S^2r^2}{4T^2t^2}\right\}$	21
$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-2S}{4Tt}+\frac{2S^2r^2}{16T^2t^2}\right\}$	17
$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-4S}{Tt}+\frac{S^2r^2}{8Tt}\right\}$	24

21 +

Your answer in Section 20,

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-S}{Tt}+\frac{S^2r^2}{4T^2t^2}\right\},$$

is correct. Now note that this expression is identical to that given for $(S/T)(\partial h/\partial t)$ in Section 20. Thus we have shown that if head is given by

$$h=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}$$

then it is true that

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

21 + —Con.

In other words, the expression

$$h=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}$$

satisfies the partial differential equation, or constitutes one particular solution to it. In fact, this expression is the solution which describes the hydraulic head in an infinite, horizontal, homogeneous, and isotropic artesian aquifer, after a finite volume of water, V, is injected suddenly at $t=0$ into a fully penetrating well of infinitesimal radius located at $r=0$, assuming that head was everywhere at the datum prior to the injection—that is, assuming h was everywhere zero prior to $t=0$.

Proof that our function is the solution corresponding to this problem requires, in addition to the demonstration that it satisfies the differential equation, proof that it satisfies the various boundary conditions peculiar to the problem. We now wish to formulate these conditions.

The charge of fluid is added to the well at the instant $t=0$. At this instant, there has been no time available for fluid to move away from the well, into the aquifer. Therefore, at all points in the aquifer except at the well (that is, except at $r=0$), the head at $t=0$ must still be zero. In the well, on the other hand, the addition of the volume of water produces an instantaneous rise in head. For a well of measurable radius, this instantaneous head buildup, Δh, would be given by

$$\Delta h=\frac{V}{A_w}=\frac{V}{\pi r_w^2},$$

where A_w is the cross-sectional area of the well bore, and r_w is the well radius. For example, if A_w is 1 square foot and we inject 1 cubic foot of water, we should observe an instantaneous rise in head of 1 foot in the well; and because head was originally at 0 (datum level), we can say that the head in the well at $t=0$ should be 1 foot. If A were 0.5 square foot, the head in the well at $t=0$ should be 2 feet; and so on.

QUESTION

For purposes of developing the boundary conditions, we have assumed the radius of our well to be infinitesimally small—that is, to approach zero. Which of the following statements describes the behavior of head at the well at $t=0$, subject to this assumption?

	Turn to Section:
head at the well will be 0 feet at $t=0$	10
head at the well will be 1 foot at $t=0$	19
head at the well will be immeasurably large —that is, infinite—at $t=0$	18

22 +

Your answer in Section 37 is not correct. The expression obtained in Section 28 for inflow minus outflow was

$$Q_1-Q_2=2\pi T\left\{r\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r}\right\}\Delta r.$$

Our expression for dV/dt was

$$\frac{dV}{dt}=S2\pi r\Delta r\frac{\partial h}{\partial t}.$$

The expression for inflow minus outflow may be equated to that for dV/dt, and the result simplified to yield the correct answer.

Return to Section 37 and choose another answer.

23+

Your answer in Section 5,

$$\frac{\partial^2 h}{\partial r^2}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\left(\frac{-2S}{4Tt}\right)+\left(\frac{-2Sr}{4Tt}\right)\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right),$$

is not correct. The rule for differentiation of a product is: first factor times derivative of second plus second factor times derivative of first. The two factors, in this case, are

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}$$

(which we have already differentiated in the question of Section 27) and

$$\frac{-2Sr}{4Tt}.$$

The first term of your answer is correct; the first factor,

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}$$

is multiplied by the derivative of the second, which is

$$\frac{-2S}{4Tt}$$

(t is simply treated as part of the constant coefficient of r, since we are differentiating with respect to r). The second term of your answer, however, is not correct; you have written the derivative of the first factor as

$$e^{-(Sr^2/4Tt)}\cdot\left(\frac{-2Sr}{4Tt}\right).$$

Compare this with the correct answer to the question of Section 27 and you will see that it does not represent the derivative of

$$\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}.$$

Return to Section 5 and choose another answer.

24+

Your answer in Section 20,

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\left\{\frac{-4S}{Tt}+\frac{S^2r^2}{8Tt}\right\},$$

is not correct. This answer contains algebraic errors, both in the addition of the two terms

$$\left(\frac{-2S}{4Tt}\right)$$

and in the multiplication of the two terms

$$\left(\frac{-2Sr}{4Tt}\right).$$

Return to Section 20 and choose another answer.

25+

Your answer in Section 15,

$$Q_1-Q_2=2\pi T\left\{\left(\frac{\partial h}{\partial r}\right)_2-\left(\frac{\partial h}{\partial r}\right)_1\right\},$$

is not correct. The expression for inflow through the inner cylindrical face was shown to be

$$Q_1=-2\pi T\left(r\frac{\partial h}{\partial r}\right)_1.$$

Applying Darcy's law in a similar fashion to the outer cylindrical face, at radius r_2, the expression for outflow through this face is found to be

$$Q_2=-2\pi T\left(r\frac{\partial h}{\partial r}\right)_2.$$

These two equations may be subtracted to obtain an expression for inflow minus outflow. The radius, r, does not disappear in this subtraction. Your answer, which does not include radius, must therefore be wrong.

Return to Section 15 and choose another answer.

26+

Your answer in Section 7,

$$\frac{\partial\left(r\frac{\partial h}{\partial r}\right)}{\partial r}=r\frac{\partial\left(\frac{\partial h}{\partial r}\right)}{\partial r}+\frac{\partial h}{\partial r}\cdot r,$$

is not correct. The derivative of a product is given by the first factor multiplied by the derivative of the second, plus the second factor multiplied by the derivative of the first. Your first term, above is correct; the first factor, r, is multiplied by the derivative of $\partial h/\partial r$, although it would be more conventional to use the second derivative notation,

$$\frac{\partial^2 h}{\partial r^2},$$

rather than

$$\frac{\partial\left(\frac{\partial h}{\partial r}\right)}{\partial r}.$$

Your second term, however, is not correct. The derivative of r with respect to r is not equal to r.

Return to Section 7 and choose another answer.

27+

Your answer in Section 37 is correct. The basic differential equation for the problem is

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

In seeking a solution to this equation, we are seeking an expression giving h as a function of r and t, such that when $\partial h/\partial r$, $\partial^2 h/\partial r^2$, and $\partial h/\partial t$ are obtained by differentiation and substituted into this equation, the equation is found to be satisfied. For example, consider the function

$$h=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}$$

in which V (as well as S and T) is constant and e is the base of natural logarithms. This happens to be an important function in the theory of well hydraulics, as we shall see; and we wish now to test it, to see whether it satisfies the above differential equation. To do this we must differentiate the expression once with respect to t and twice with respect to r; these operations are not difficult if the rules of differentiation are applied carefully. First we will differentiate with respect to r; in doing so, we treat t as a constant, so that the factor $V/(4\pi Tt)$ becomes simply a constant coefficient. In the exponent, as well, the term $-(S/4Tt)$ may be considered a constant coefficient of r^2; and the problem is essentially one of finding the derivative of $e^{-(S/4Tt)r^2}$ and multiplying this by the constant factor $V/(4\pi Tt)$. The derivative of a function e^u with respect to a variable r is given simply by $e^u\cdot(du/dr)$. Here, u is the term $-(S/4Tt)r^2$.

QUESTION

Following the procedure outlined above, which of the following expressions is found for $\partial h/\partial r$?

	Turn to Section:
$\dfrac{\partial h}{\partial r}=e^{-(Sr^2/4Tt)}\cdot\left(\dfrac{-2Sr}{4Tt}\right)$	4
$\dfrac{\partial h}{\partial r}=\dfrac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}\cdot\left(\dfrac{-2Sr}{4Tt}\right)$	5
$\dfrac{\partial h}{\partial r}=\dfrac{V}{4\pi Tt}e^{-(Sr^2/4Tt)}$	2

28+

Your answer in Section 7,

$$\frac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r}=r\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r},$$

is correct. Our expression for

$$\left(r\frac{\partial h}{\partial r}\right)_2-\left(r\frac{\partial h}{\partial r}\right)_1$$

may therefore be written

$$\left(r\frac{\partial h}{\partial r}\right)_2-\left(r\frac{\partial h}{\partial r}\right)_1$$

$$=\frac{\partial\left(r\dfrac{\partial h}{\partial r}\right)}{\partial r}\Delta r=\left\{r\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r}\right\}\Delta r.$$

Our expression for inflow minus outflow therefore becomes

(*continued on next page*)

28 + —Con.

$$Q_1 - Q_2 = 2\pi T\left\{\left(r\frac{\partial h}{\partial r}\right)_2 - \left(r\frac{\partial h}{\partial r}\right)_1\right\}$$

$$= 2\pi T\left\{r\frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{\partial r}\right\}\Delta r.$$

As before, we wish to equate this expression for inflow minus outflow to the rate of accumulation of water in storage in our element. The surface area of the cylindrical element is given approximately by

$$A = 2\pi r \Delta r.$$

The term $2\pi r$ is the perimeter of a circle taken along the midradius of the element; multiplication by the radial width, Δr gives the surface area, or base area, of the cylindrical shell.

QUESTION

Using this expression for the surface area of the cylindrical element, and letting $\partial h/\partial t$ denote the time rate of head buildup in the element, which of the following expressions is obtained for the rate of accumulation of water in storage in the element?

	Turn to Section:
$\frac{dV}{dt} = S2\pi r\Delta r\frac{\partial h}{\partial t}$	37
$\frac{dV}{dt} = S\pi r^2\frac{\partial h}{\partial t}$	12
$\frac{dV}{dt} = \frac{S\frac{\partial h}{\partial t}}{2\pi r\Delta r}$	16

29 +

Your answer in Section 18 is not correct. The behavior we are trying to describe is the disappearance of the effect of injection, at great radial distances from the well. The answer which you chose describes head, h, as going to infinity, rather than disappearing; and it describes a restriction on h with time, rather than with distance.

Return to Section 18 and choose another answer.

30 +

Your answer in Section 15,

$$Q_1 - Q_2 = 2\pi T\left(r\frac{\partial h}{\partial r}\right)_1 \cdot \left(r\frac{\partial h}{\partial r}\right)_2,$$

is not correct. We established in Sections 1 and 15 that inflow through the inner cylindrical face of the element is given by Darcy's laws as

$$Q_1 = -2\pi T\left(r\frac{\partial h}{\partial r}\right)_1.$$

Using a similar approach, we can show that outflow through the outer cylindrical face is given by

$$Q_2 = 2\pi T\left(r\frac{\partial h}{\partial r}\right)_2.$$

These two equations can be subtracted to obtain an expression for inflow minus outflow for the cylindrical element.

Return to Section 15 and choose another answer.

31 +

Your answer in Section 35,

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}\cdot\frac{Sr^2}{4Tt^2}+e^{-(Sr^2/4Tt)}\cdot\left(\frac{-V}{4\pi Tt^2}\right),$$

is not correct. Application of the product rule—first factor times derivative of second plus second factor times derivative of first—is correct; but your expression for the time derivative of $e^{-(Sr^2/4Tt)}$ is not correct.

Recall that the derivative of an exponential, e^u, with respect to t is given by $e^u du/dt$. Letting u represent $-(Sr^2/4Tt)$, your answer gives only $\partial u/\partial t$ in the place where it should give

$$e^u\frac{\partial u}{\partial t}.$$

Return to Section 35 and choose another answer.

32 +

Your answer in Section 37 is not correct. In Section 28, we saw that the expression for inflow minus outflow could be written

$$Q_1-Q_2=2\pi T\left\{r\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r}\right\}\Delta r$$

while the expression we obtained for dV/dt was

$$\frac{dV}{dt}=S2\pi r\Delta r\frac{\partial h}{\partial t}.$$

If we equate the terms

$$2\pi T\left\{r\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r}\right\}\Delta r$$

and

$$S2\pi r\Delta r\frac{\partial h}{\partial t}$$

and then divide through the resulting equation by

$$2\pi Tr\Delta r,$$

we obtain the correct answer to the question of Section 37.

Return to Section 37 and choose another answer.

33 +

Your answer in Section 18, $h\to 0$ as $r\to\infty$ is correct. From a mathematical point of view, we should perhaps have used, instead, the condition that $(\partial h/\partial r)\to 0$ as $r\to\infty$. This condition is required as r increases toward infinity, because the cross sectional area of flow within the aquifer—a cylindrical area coaxial with the well—expands toward infinity. Thus if we were to apply Darcy's law to determine the flow of the injected water away from the well, we would obtain the result that this flow increases toward an infinite value with increasing distance from the well, unless we postulated that the head gradient, $\partial h/\partial r$, decreased toward zero with increasing r. However, the condition that h approaches a constant, 0, as $r\to\infty$ implies that $\partial h/\partial r$ must also approach zero as r increases; and it is a somewhat easier condition to establish.

Our task, then, is to show that the function

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

satisfies this condition—that is, we must test

33 + —Con.

this function to see whether its value approaches zero as r approaches infinity. It is easy to show that for any finite value of time the condition is satisfied. However, we are also interested in what happens as t approaches infinity along with r—that is, we would like our condition to be satisfied for all times, even those immeasurably large. For this reason, it is convenient to use the the series expansion form given in Section 18; that is we use

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

$$=\frac{V}{4\pi Tt+r^2S\pi+\dfrac{r^4S^2\pi}{4Tt\cdot 2!}+\dfrac{r^6S^3\pi}{16T^2t^2\cdot 3!}}$$

In order that the fraction on the right approach zero, it is sufficient that any one of the individual terms in the denominator becomes infinite. If r and t both approach infinity, the first two terms clearly become infinite; in fact, the remaining terms become infinite as well, although we need not show this. If one term is infinite, the entire denominator is infinite, and the fraction is zero. For a finite value of t, all terms except the first clearly become infinite as $r\to\infty$, and again the expression as a whole tends to zero. Thus the expression

$$\frac{V}{4\pi Tt}e^{-(r^2S/4Tt)}$$

satisfies the condition of tending to zero as $r\to\infty$, for any value of time. Again, this can be demonstrated by extending the plots described in Section 18 to large values of r.

We could also add the condition that h must approach zero as time becomes infinite, everywhere in the aquifer—that is, that the effect of the injection must eventually die out with time everywhere throughout the aquifer, since we are injecting a finite volume of water into an aquifer which is assumed to be infinite in extent. We have just shown that h approaches zero at infinite time, as r also becomes infinite; we need only show that this behavior holds when r is finite. We will show this through direct use of the function, although it is also evident using the series expansion form. As t becomes infinitely large the factor

$$\frac{V}{4\pi Tt}$$

must approach zero; the factor

$$-e^{-(r^2S/4Tt)}$$

which is equivalent to

$$\frac{1}{e^{(r^2S/4Tt)}}$$

must approach the value

$$\frac{1}{e^{r^2S/\infty}}$$

or

$$\frac{1}{e^0},$$

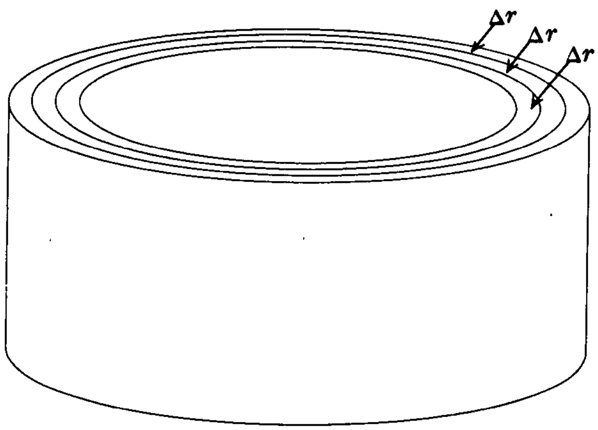
if r is finite. But e^0 is simply 1, so that the product

$$\frac{V}{4\pi Tt}\cdot e^{-(r^2S/4Tt)}$$

must approach zero as t becomes infinitely large, at any finite value of r.

We now consider the last condition which our function should satisfy. In the sketch, the aquifer has been divided into cylindrical elements of radial width Δr, coaxial with the well. At any given time t after injection, the injected volume of fluid, V, is distributed in some way among these cylindrical elements.

33 + —Con.



We assumed head to be at the datum, or zero, prior to injection, so that h actually represents only the head increase due to the injection. From the definition of storage coefficient, the quantity of the injected fluid contained within a given cylindrical element will be given by

$$\Delta V = S \cdot h_{r,t} \cdot 2\pi r \Delta r$$

where r is the median radius of the element, so that $2\pi r \Delta r$ is the base area of the element; $h_{r,t}$ gives the average head in the element (that is, at the radius r) at the time in question; and S is the storage coefficient. (Recall the definition of storage coefficient—the volume in storage is the product of storage coefficient, head, and base area.) Now if we sum the volumes in storage in every cylindrical element in the aquifer, the total must equal the injected volume, V, at any time after injection. That is,

$$V = \Sigma \Delta V = \Sigma S \cdot h_{r,t} \cdot 2\pi r \Delta r$$

where the summation is carried out over all of the cylindrical elements in the aquifer. Again, it should be kept in mind that $h_{r,t}$ represents only the head increase associated with the injection, so that its use in the storage equation leads only to the volume of water injected, not to the total volume in storage. Now since we are dealing with a continuous system, we replace the summation in the above equation by an integration. That is, we let the width of each element become infinitesimally small, denoting it dr, so that the number of elements becomes infinitely great; and we rewrite our equation as

$$V = \int_{r=0}^{r=\infty} S \cdot h_{r,t} \cdot 2\pi r dr.$$

The limits of integration extend from $r=0$ to $r=\infty$, indicating that the cylindrical elements extend over the entire aquifer. This equation then is the final condition which our function should satisfy if it is in fact the solution we are seeking.

QUESTION

How do you think our proposed solution should be tested to see if it satisfies this boundary condition?

Turn to Section:

The integration indicated in the equation should be carried out. The result should equal

$$\frac{V}{4\pi T t} e^{-(r^2S/4Tt)}.$$ 11

The expression

$$\frac{V}{4\pi T t} e^{-(r^2S/4Tt)}$$

should be substituted for

$$2\pi r$$

in the equation, and the integration should be carried out; the result should be

$$2\pi S.$$ 14

The expression

$$\frac{V}{4\pi T t} e^{-(r^2S/4Tt)}$$

should be substituted for

$$h_{r,t}$$

in the equation, and the integration should be carried out; the result should equal

$$V.$$ 13

34 +

Your answer in Section 1,

$$Q_1 = -K\pi r_1^2 \left(\frac{\partial h}{\partial r}\right)_1$$

is not correct. Darcy's law states that flow is given by the product of hydraulic conductivity, head gradient in the direction of flow, and cross-sectional area normal to the direction of flow. In this problem as in the steady flow to a well treated in Part III, the direction of flow is the radial, or r, direction. An area which is everywhere normal to the radial coordinate would be a cylindrical area, coaxial with the well. That is, the flow area that we require here is a cylindrical area—in particular, the inner face of the cylindrical prism shown in Section 1. The area of a cylinder is given by the product of its height and its perimeter.

Return to Section 1 and choose another answer.

35 +

Your answer in Section 5,

$$\frac{\partial^2 h}{\partial r^2} = \frac{V}{4\pi Tt}\left\{e^{-(Sr^2/4Tt)} \cdot \left(\frac{-2S}{4Tt}\right) + \left(\frac{-2Sr}{4Tt}\right) \cdot e^{-(Sr^2/4Tt)} \cdot \left(\frac{-2Sr}{4Tt}\right)\right\},$$

is correct. We now wish to differentiate the equation

$$h = \frac{V}{4\pi Tt} \cdot e^{-(Sr^2/4Tt)}$$

with respect to time, to obtain an expression for $\partial h/\partial t$. In doing this, we consider r to be a constant, and treat our expression as the product of the two functions of t,

$$\frac{V}{4\pi Tt}$$

and

$$e^{-(Sr^2/4Tt)}.$$

The derivative of

$$\frac{V}{4\pi Tt}, \text{ or } \frac{V}{4\pi T} \cdot t^{-1}$$

with respect to t is

$$-\frac{V}{4\pi T} \cdot t^{-2}, \text{ or } \frac{-V}{4\pi Tt^2}.$$

To differentiate

$$e^{-(Sr^2/4Tt)}$$

we again apply the rule

$$\frac{de^u}{dt} = e^u \frac{du}{dt},$$

where u is

$$\frac{-Sr^2}{4Tt}, \text{ or } \frac{-Sr^2}{4T} \cdot t^{-1},$$

and its derivative with respect to t is

$$\frac{Sr^2}{4T} \cdot t^{-2}, \text{ or } \frac{Sr^2}{4Tt^2}.$$

QUESTION

Applying the rule for differentiation of a product, together with the above results, which of the following expressions is obtained for $\partial h/\partial t$?

35 + —Con.

Turn to Section:

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\cdot\left(\frac{Sr^2}{4Tt^2}\right)$$ 3

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}\cdot e^{-(Sr^2/4Tt)}\left(\frac{Sr^2}{4Tt^2}\right)+e^{-(Sr^2/4Tt)}\cdot\left(\frac{-V}{4\pi Tt^2}\right)$$ 20

$$\frac{\partial h}{\partial t}=\frac{V}{4\pi Tt}\cdot\frac{Sr^2}{4Tt^2}+e^{-(Sr^2/4Tt)}\left(\frac{-V}{4\pi Tt^2}\right)$$ 31

36 +

Your answer in Section 1,

$$Q_1=\frac{-Kb\left(\frac{\partial h}{\partial r}\right)_1}{2\pi r_1},$$

is not correct. Darcy's law tells us that flow is given by the product of hydraulic conductivity, head gradient in the direction of flow, and cross-sectional area normal to the direction of flow. In this case, as in the steady state flow to a well in Part III, the direction of flow is the radial direction and the cross-sectional area normal to the flow is a cylindrical surface—the inner surface of the cylindrical shell shown in Section 1. In your answer, however, there is no factor representing the area of this surface. The height of the cylinder, which is b, appears in the numerator of your answer; its perimeter, which is $2\pi r_1$, appears in the denominator of the answer which you chose.

Return to Section 1 and choose another answer.

37 +

Your answer in Section 28,

$$\frac{dV}{dt}=S2\pi r\Delta r\frac{\partial h}{\partial t},$$

is correct. As before, we will next use the equation of continuity to link the storage and flow equations.

QUESTION

If the expression obtained for inflow minus outflow is equated to that given above for rate of accumulation in storage, which of the following equations may be obtained?

Turn to Section:

$$r\frac{\partial^2 h}{\partial r^2}+\frac{1}{2\pi r}\frac{\partial h}{\partial r}=S\frac{\partial h}{\partial t}$$ 22

$$2\pi T\Delta r\left\{\frac{\partial^2 h}{\partial r^2}+\frac{\partial h}{\partial r}\right\}=S\frac{\partial h}{\partial t}$$ 32

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t}$$ 27

Development of Additional Solutions by Superposition

The differential equation

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t}$$

is linear in h; that is, h and the various derivatives of h occur only in the first power—they are not squared, cubed, or raised to any power except 1, in any term of the equation. Equations of this type have the property that solutions corresponding to two individual disturbances may be added to obtain a new solution describing the effect of the two disturbances in combination. This is termed superposition of solutions; it is a technique which is often used intuitively by hydrologists—for example when calculating the drawdown produced by several wells, by adding drawdowns calculated for individual operation.

The solution obtained in the preceding programed instruction was developed for an injection of fluid at $t=0$. If the injection does not occur at $t=0$, the term t in the solution is simply replaced by Δt, the time interval between the injection and the instant of head measurement. For example, if the injection occurs at time t', and the head change due to this injection is measured at some later time t, the interval $t-t'$ is used in the solution in place of t, giving

$$h_{r,t}=\frac{V}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}.$$

Now suppose two injections occur, one at t_1' and one at t_2', and the head is measured at some time t following both injections. Using superposition, the head change due to the combined disturbances is

$$h_{r,t}=\frac{V_1}{4\pi T(t-t_1')}\cdot e^{-\left(\frac{r^2S}{4T(t-t_1')}\right)}+\frac{V_2}{4\pi T(t-t_2')}\cdot e^{-\left(\frac{r^2S}{4T(t-t_2')}\right)}$$

where V_1 is the volume injected at t_1' and V_2 is the volume injected at t_2'.

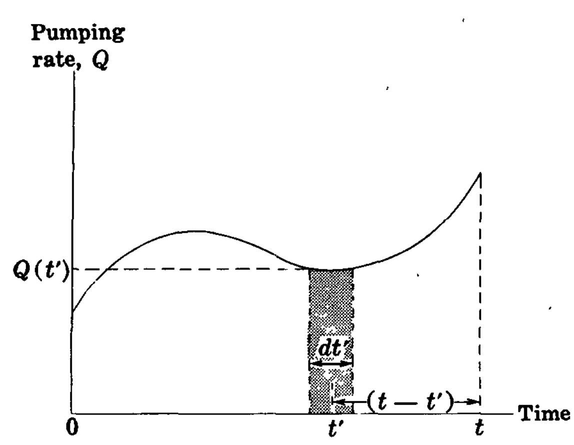
If we consider removal of a volume of water from the well, rather than injection, we need only introduce a change of sign, taking V as negative. For example, if a bailerfull of water is removed at $t=t_1'$, the head change at time t, due to this removal is

$$h_{r,t}=\frac{-V_1}{4\pi T(t-t_1')}\cdot e^{-\left(\frac{r^2S}{4T(t-t_1')}\right)}$$

where V_1 is the volume removed by the bailer. If the well is bailed repeatedly, as may happen during completion, the head change due to bailing is obtained by superposing the disturbances due to each individual withdrawal:

$$h_{r,t} = -\frac{V_1}{4\pi T(t-t_1')} \cdot e^{-\left(\frac{r^2S}{4T(t-t_1')}\right)} - \frac{V_2}{4\pi T(t-t_2')} \cdot e^{-\left(\frac{r^2S}{4T(t-t_2')}\right)} - \frac{V_3}{4\pi T(t-t_3')} \cdot e^{-\left(\frac{r^2S}{4T(t-t_3')}\right)} * * * - \frac{V_n}{4\pi T(t-t_n')} \cdot e^{-\left(\frac{r^2S}{4T(t-t_n')}\right)}$$

where t is the time at which h is measured; t_1', t_2', t_3', * * * t_n' are the times at which the individual withdrawals are made; and V_1, V_2, V_3, * * * V_n are the volumes removed by the bailer in the successive withdrawals. The "bailer method" of determining transmissivity from the residual drawdown of a well that has been bailed was developed from this equation (Skibitzke, 1963).



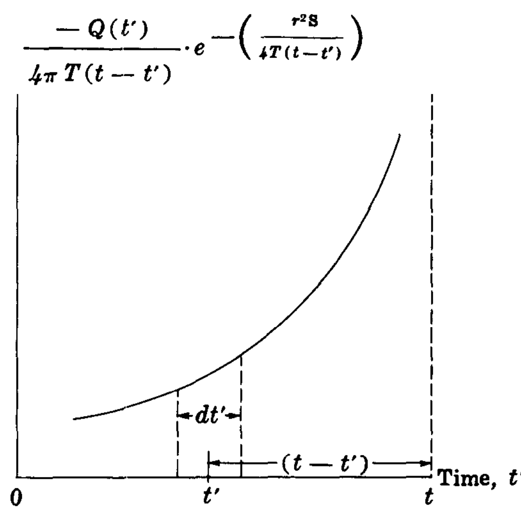
Now suppose a well is pumped continuously during the time interval from zero to t, and we wish to know the head change at time t due to this continuous withdrawal. The rate of pumping, in volume of water per unit time, may vary from one instant to the next. The figure shows a plot of pumping rate verus time for a hypothetical case. Pumping starts at time = 0 and extends to time = t, the instant at which we wish to know the head change. We consider first the head change at t due to the action of the pump at one particular instant, t', during the course of pumping. We consider an infinitesimal time interval, dt', extending to either side of the instant t'; the average rate of pumping during this interval is denoted $Q(t')$. The volume of water withdrawn from the well during the interval is the product of the pumping rate, $Q(t')$, and the time interval, dt'; that is,

$$-V = -Q(t')\,dt'.$$

Again negative signs are used to indicate withdrawal as opposed to injection. The product $Q(t')dt'$ is equal to the area of the shaded element in the graph shown in the preceding figure; the height of this element is $Q(t')$, and its width is dt'. The time interval betwen the instant of withdrawal and the instant of head measurement is $t-t'$. Using the solution obtained in the programed instruction for the head change due to instantaneous withdrawal of a volume of water, the head change at time t due to the withdrawal at t' is given by

$$\frac{-V}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)} = \frac{-Q(t')\,dt'}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}$$

The total head change at t, due to the continuous withdrawal from zero to t, is obtained through superposition, by adding the head changes due to the instantaneous withdrawals throughout the interval from zero to t.



The figure shows a graph in which, instead of plotting only discharge versus time, we plot the entire function

$$\frac{-Q(t')}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}$$

versus time. The area of the element at t' is now

$$\frac{-Q(t')}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}\cdot dt'$$

—thus it is just equal in magnitude to the head change at t, caused by the withdrawal at t'. If elements of the type shown in the figure are constructed all along the time axis, from zero to t, the area of each element will give the head change at t due to operation of the pump during the time interval represented by the element; the total head change at t due to all of the instantaneous withdrawals throughout the interval from zero to t will therefore be equal to the sum of these areas, or the total area under the curve from zero to t. This total area is the integral of the function

$$\frac{-Q(t')}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}$$

over the interval from zero to t, that is, the total head change is given by

$$h=\int_{t'=0}^{t'=t}\frac{-Q(t')}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}dt'.$$

It should be noted that we are now using t' to denote the time *variable* or variable of integration, rather than to specify one particular instant. The function being integrated involves the difference, $t-t'$, between the upper limit of integration and the variable of integration. Evaluation of the integral will yield a function of the upper limit, t, and of r; that is, the head change due to the pumping will be specified as a function of r and of t (the time of head measurement.)

For the particular case when the rate of discharge is a constant, Q, the integral equation can be transformed directly into a form suitable for computation. We have

$$h=\int_{t'=0}^{t'=t}\frac{-Q}{4\pi T(t-t')}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}dt'.$$

The term $-Q/4\pi T$ is a constant and may be taken outside the integral, giving

$$h=\frac{-Q}{4\pi T}\int_{t'=0}^{t'=t}\frac{1}{t-t'}\cdot e^{-\left(\frac{r^2S}{4T(t-t')}\right)}dt'.$$

We introduce the algebraic change of variable,

$$\psi=\frac{r^2S}{4T(t-t')}.$$

We differentiate this expression with respect to t', treating t, at this stage, as a constant; this gives

$$\frac{d\psi}{dt'}=\frac{r^2S\cdot 4T}{(4T(t-t'))^2}=\frac{r^2S}{4T(t-t')}\cdot\frac{1}{t-t'}$$

$$=\frac{r^2S}{4T(t-t')}\left(\frac{\frac{r^2S}{4T(t-t')}}{\frac{r^2S}{4T}}\right)=\frac{\psi^2}{\frac{r^2S}{4T}}.$$

Therefore

$$d\psi=\frac{\psi^2}{\frac{r^2S}{4T}}\cdot dt'$$

and

$$dt'=\frac{r^2S}{4T}\frac{d\psi}{\psi^2}.$$

The value of ψ corresponding to the upper limit of integration, $t'=t$, is

$$\psi_t=\frac{r^2S}{4T(t-t)}=\infty.$$

While the value of ψ corresponding to the lower limit of integration, $t'=0$, is

$$\psi_0=\frac{r^2S}{4T(t-0)}=\frac{r^2S}{4Tt}.$$

We now return to our integral equation and substitute ψ for

$$\frac{r^2S}{4T(t-t')};$$

$$\frac{r^2S}{4T}\cdot\frac{d\psi}{\psi^2}$$

for

$$dt';$$

and the values obtained above for the limits of integration. This gives

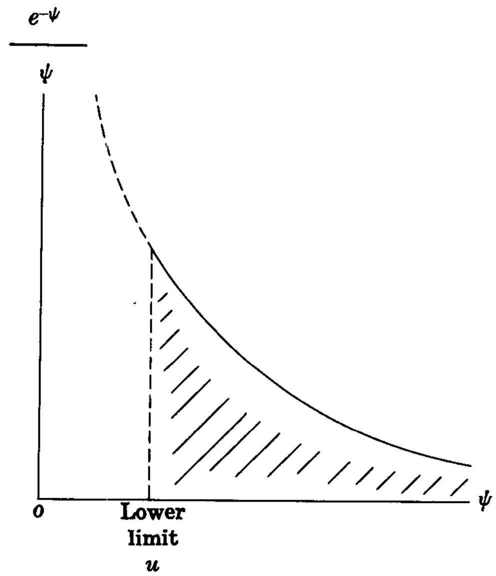
$$h=\frac{-Q}{4\pi T}\int_{\frac{r^2S}{4Tt}}^{\infty}\frac{1}{t-t'}\cdot e^{-\psi}\cdot\frac{r^2S}{4T}\cdot\frac{d\psi}{\psi^2}.$$

But since

$$\frac{1}{t-t'}\cdot\left(\frac{r^2S}{4T}\right)=\psi,$$

the above integral becomes

$$h=\frac{-Q}{4\pi T}\int_{\frac{r^2S}{4Tt}}^{\infty}\frac{e^{-\psi}}{\psi}\,d\psi.$$



This integral is called the exponential integral. It is a function of its lower limit, as suggested by the figure, which shows a graph of the function $e^{-\psi}/\psi$ versus ψ. The area under this graph is equal to the value of the integral. The upper limit is infinite, and the function $e^{-\psi}/\psi$ approaches zero as ψ becomes infinite; the area under the curve, or the value of the integral, depends only upon the point where the lower limit is taken—that is, upon the value of $r^2S/4Tt$. This term is often denoted u in the literature, so that the equation for head change is often written

$$h=\frac{-Q}{4\pi T}\int_{u}^{\infty}\frac{e^{-\psi}}{\psi}\,d\psi$$

where

$$u=\frac{r^2S}{4Tt}.$$

It can be shown that the above integral is equal to an infinite series involving the lower limit. Specifically,

Values of the integral for various values of the lower limit have been computed, using this series, and tabulated. In the hydrologic literature, the value of the integral is commonly referred to as $W(u)$ or "well function of u." Tables of $W(u)$ versus u are available in the reference by Ferris, Knowles, Brown, and Stallman (1962) and in numerous other references. In the forms presented above, the equations yield the head change, or simply the head, assuming h was zero prior to pumping. If head was at some other constant level, h_0, prior to pumping, the expressions are still valid for head change, $h-h_0$. That is, we have

$$h-h_0=\frac{-Q}{4\pi T}\int_{u}^{\infty}\frac{e^{-\psi}}{\psi}\,d\psi=\frac{-Q}{4\pi T}\cdot W\,(u)$$

where

$$u=\frac{r^2S}{4Tt}$$

or in terms of drawdown, h_0-h, we have

$$s=h_0-h=\frac{Q}{4\pi T}\int_{u}^{\infty}\frac{e^{-\psi}}{\psi}\,d\psi=\frac{Q}{4\pi T}\cdot W\,(u)$$

The result we have obtained here is known as the Theis equation, after C. V. Theis who first applied it in hydrology (Theis, 1935). An excellent discussion of the significance of this equation in hydrology is given in another paper by Theis (1938).

It was recognized by Cooper and Jacob (1946) that at small values of u, (that is, at large values of t), the terms following $\ln(u)$ in the series expansion for

$$\int_{u}^{\infty}\frac{e^{-\psi}}{\psi}\,d\psi=-0.5772-\ln(u)+u-\frac{u^2}{2\cdot 2!}+\frac{u^3}{3\cdot 3!}-\frac{u^4}{4\cdot 4!}+{***}$$

$$\int_{\mu}^{\infty} \frac{e^{-\psi}}{\psi} d\psi$$

become negligibly small. In this condition the value of the integral is given simply by

$$-0.5772-\ln(u),$$

or

$$-0.5772-\ln\left(\frac{r^2S}{4Tt}\right).$$

The sign of the logarithmic term may be changed by inverting the expression in brackets,

$$-\ln\left(\frac{r^2S}{4Tt}\right)=\ln\left(\frac{4Tt}{r^2S}\right)$$

and the constant, 0.5772, may be expressed as the natural logarithm of another constant,

$$0.5772=\ln\left(\frac{4}{2.25}\right)$$

so that

$$-0.5772-\ln\left(\frac{r^2S}{4Tt}\right)=\ln\left(\frac{4Tt}{r^2S}\right)-\ln\left(\frac{4}{2.25}\right)$$
$$=\ln\left(\frac{2.25Tt}{r^2S}\right)=2.3\log_{10}\left(\frac{2.25Tt}{r^2S}\right).$$

Thus when pumping has continued for a sufficient length of time so that u, or $r^2S/4Tt$, is small we may write

$$s=\frac{Q}{4\pi T}\int_{u}^{\infty}\frac{e^{-\psi}}{\psi}d\psi\approx\frac{2.3Q}{4\pi T}\log_{10}\left(\frac{2.25\cdot Tt}{r^2S}\right).$$

This is the modified nonequilibrium formula, which forms the basis of the "semilog plot" techniques often used by hydrologists in the analysis of pumping test data. These techniques are generally applied for values of u less than 0.01.

The Theis equation and the modified nonequilibrium formula are extremely useful hydrologic tools, provided they are used within the limits of application established by the assumptions made in their derivation. Before leaving this subject, we will briefly review the assumptions that have been accumulated during the course of the derivation. We first developed the equation

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t}$$

by assuming that:

1. The aquifer was confined;
2. There was no vertical flow;
3. All flow was directed radially toward (or away from) the origin;
4. S and T were constant—that is, the aquifer was homogeneous and isotropic;
5. There was no areal recharge applied to the aquifer

In writing the solution corresponding to instantaneous discharge or input of a volume of water, V, we added the assumptions that:

6. The aquifer was infinite in extent;
7. There was no lateral discharge or recharge except at the well
8. The head was uniform and unchanging throughout the aquifer prior to $t=0$.
9. All of the injected water was taken into storage (or conversely, all discharged water was derived from storage).
10. The well was of infinitesimal radius.

Finally, when we integrated the above solution to obtain the continuous discharge solution

$$s=h_0-h=\frac{Q}{4\pi T}\int_{\frac{r^2S}{4Tt}}^{\infty}\frac{e^{-\psi}}{\psi}d\psi$$

we added the condition that

11. The discharge, Q, was constant throughout the duration of pumping.

These assumptions should be kept in mind whenever the Theis equation is applied. The assumption that all flow is lateral implies that the well must fully penetrate the aquifer and that the aquifer is horizontal.

If the semilog approximation is used, we add the assumption that the time is great enough and radius small enough that the term $r^2S/4Tt$ is less than 0.01, and the later terms in the series expression for the integral can therefore be neglected.

The Theis equation was the first equation to describe flow of water to a well under nonequilibrium conditions. In subsequent work, Papadopulos and Cooper (1967 have accounted for the effects of a finite well radius; Jacob (1963) and several other writers have examined the problem of discharge from partially penetrating wells; Stallman (1963a), Lang (1963), and numerous other investigators have utilized image theory to account for lateral aquifer boundaries; Jacob and Lohman (1952) have analyzed discharge at constant drawdown, rather than at constant rate; numerous writers, including in particular Jacob (1946), Hantush (1959, 1960 1967a 1967b) and Hantush and Jacob (1955) have treated the problem of discharge from an aquifer replenished by vertical recharge through overlying and underlying strata; and several writers, including Boulton (1954), have attacked the general problem of three-dimensional flow to a well. Weeks (1969) has applied various aspects of the theory of flow toward wells to the problem of determining vertical permeability from pumping test analysis.