Techniques of Water-Resources Investigations
of the United States Geological Survey

Chapter B2

INTRODUCTION TO GROUND-WATER HYDRAULICS

A Programed Text for Self-Instruction

By Gordon D. Bennett

Book 3

APPLICATIONS OF HYDRAULICS

Part V. Unidirectional Nonequilibrium Flow

Introduction

In Part V, our purpose is to develop the differential equation for a problem of nonequilibrium flow. To do this, we utilize the storage equation,

$$\frac{dV}{dt} = SA\frac{dh}{dt},$$

developed in Part IV, and we utilize Darcy's law. These two relations are linked by means of a relation called the equation of continuity, which is a statement of the principle of conservation of mass.

In Part VI we will develop the same type of equation in polar coordinates and will discuss a solution to this equation for a particular flow problem. In the course of working through Parts V and VI, the reader may realize that the relations describing the storage and transmission of ground water can be combined to develop the differential equations for many other types of flow; and that solutions to these equations can be developed for a variety of field problems.

Before the start of the program of Part V, there is a brief discussion, in text form, of the significance of partial derivatives, their use in ground-water equations, and in particular their use in a more general form of Darcy's law. This form of Darcy's law was introduced in the text-format discussion at the end of Part II. The discussion here is intended primarily for readers who may not be accustomed to using partial derivatives and vector notation. It may be omitted by readers conversant with these topics. This discussion is not intended as a rigorous treatment of partial differentiation. Readers who are not familiar with the subject may wish to review such a treatment in any standard text of calculus.

Partial derivatives in ground-water flow analysis

When a dependent variable varies with more than one independent variable, the partial derivative notation is used. A topographic map, for example, may be considered a representation of a dependent variable (elevation) which is a function of two independent variables—the two map directions, which we will call x and y, as shown in figure *i*. If elevation is denoted E, each contour on the map represents a curve in the x–y plane along which E has some constant value. In general, if we move in the x direction, we will cross elevation contours—that is, E will change. Let us say that if we move a distance Δx parallel to the x axis, E is observed to change by an amount ΔE_x. We may form a ratio, $\Delta E_x/\Delta x$, of this change in elevation to the length of the x interval in which it occurs. If the interval Δx becomes vanishingly small, this ratio is designated $\partial E/\partial x$ and is termed the partial derivative of E with respect to x. $\partial E/\partial x$ is actually the slope of a plot of E versus x, at the point under consideration, or the slope of a tangent to this plot, as shown in figure *i*. Note that in obtaining $\partial E/\partial x$ we move parallel to the x axis—that is, we hold y constant, considering only the variation in E due to the change in x.

Similarly, if we move a small distance, Δy. parallel to the y axis, E will again change by some small amount, ΔE_y. We again form a ratio, $\Delta E_y/\Delta y$; if the distance taken along the y axis is vanishingly small, this ratio is designated $\partial E/\partial y$ and is termed the partial derivative of E with respect to y. Note that this time we have moved parallel to the y axis; in effect we have held x constant and isolated the variation in E due to the change in y alone.

If it happened that land surface varied so regularly over the map area that we could actually write a mathematical expression giving elevation, E, as a function of x and y, then we could compute $\partial E/\partial x$ simply by differentiating this expression with respect to x, treating y as a constant. Similarly, we could compute $\partial E/\partial y$ by differentiating the expression with respect to y, treating x as a constant. For example, suppose that after studying the contour map, we decide that elevation can be expressed approximately as a function of x and y by the equation

$$E=5x^2+10y+20.$$

Differentiating this equation with respect to x, treating y as a constant, gives

$$\frac{\partial E}{\partial x}=10x.$$

We could, therefore, compute $\partial E/\partial x$ at any point by substituting the x-coordinate of that point into the above equation. Differentiating the equation with respect to y, treating x as a constant, gives

$$\frac{\partial E}{\partial y}=10,$$

indicating that $\partial E/\partial y$ has the same value, 10, at all points of the map. In this example, $\partial E/\partial x$ turned out to be independent of y and $\partial E/\partial y$ turned out to be independent of both x and y. In general, however, $\partial E/\partial x$ may depend on both x and y, and $\partial E/\partial y$ may also depend on both x and y. For example, if E were described by the equation

$$E=5x^2+5y^2+8xy+20,$$

differentiation with respect to x would give

$$\frac{\partial E}{\partial x}=10x+8y$$

while differentiation with respect to y would give

$$\frac{\partial E}{\partial y}=10y+8x.$$

In the topographic-map example, $\partial E/\partial x$ and $\partial E/\partial y$ are space derivatives—that is, each describes the variation of E in a particular direction in space. In the discussion given in this chapter we will use the space derivative of head, $\partial h/\partial x$, giving the change in hydraulic head with respect to distance in the x direction. In addition, however, we will use the time derivative of head, $\partial h/\partial t$, giving the change in head with respect to time, if position is held fixed. $\partial h/\partial t$ is a partial derivative, just as is $\partial h/\partial x$, and it is computed according to the same rules, by considering all independent variables except t to be constant. We could in fact make a "map" of the variation of head with respect to distance and time by laying out coordinate axes marked x and t, and drawing contours of equal h in this x, t plane. The discussion given for directional derivatives in the topographic-map example could then be applied to $\partial h/\partial t$ in this example.

The partial derivative of head with respect to distance, $\partial h/\partial x$, gives the slope of the potentiometric surface in the x direction at a given point, x, and time, t. This is illustrated in figure *ii*. If x or t are varied, then in general $\partial h/\partial x$ will vary, since the slope of the potentiometric surface changes, in general, both with position and with time.

The partial derivative of head with respect to time, $\partial h/\partial t$, gives the time rate at which water level is rising or falling—that is, the slope of a hydrograph—at a given point, x, and time, t. This is shown in figure *iii*. Again, if x or t are varied, then in general $\partial h/\partial t$ will vary. In other words, $\partial h/\partial x$ is a function of both x and t, and $\partial h/\partial t$ is also a function of both x and t, in the general case.

Physically, $\partial h/\partial x$ may be thought of as the slope of the potentiometric surface which

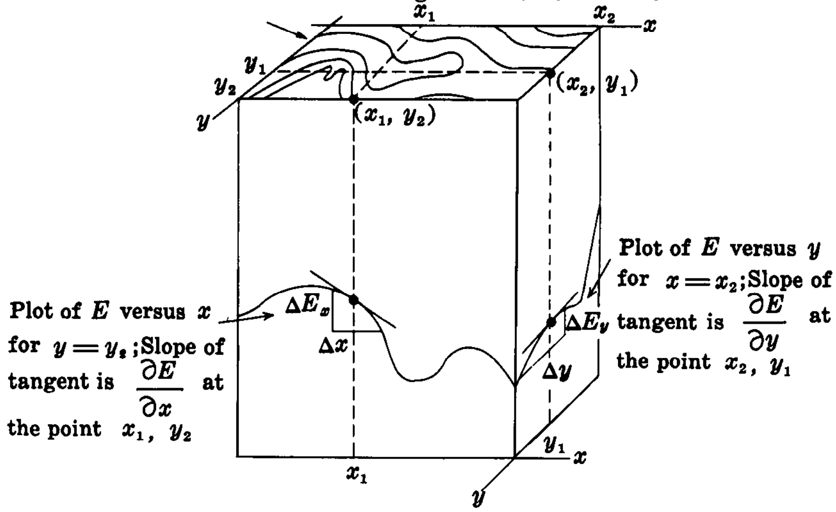


fig. *i*

will be observed if time is suddenly frozen at some value. If an expression is given for h, as a function of x and t, $\partial h/\partial x$ can be calculated by differentiating this expression with respect to x, treating t as a constant. In the same way, $\partial h/\partial t$ may be visualized as the slope of a hydrograph recorded at a particu-

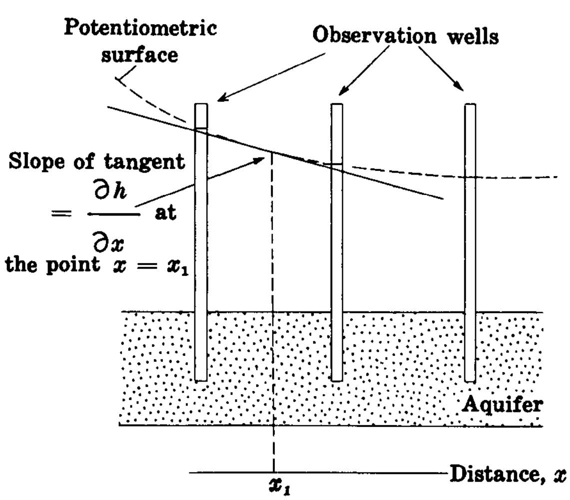


fig. *ii*

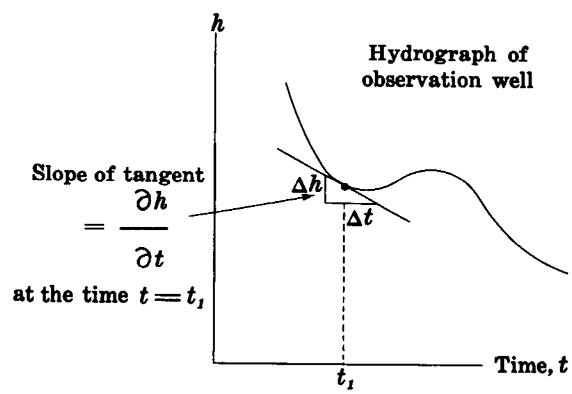


fig. *iii*

lar location (x value). If h is given as a function of x and t, an expression for $\partial h/\partial t$ may be obtained by differentiating with respect to t, treating x as if it were a constant.

In the discussion in Part V the problem is restricted to only one space derivative, $\partial h/\partial x$, and the time derivative. In the general case, we would have to consider all three space derivatives—$\partial h/\partial x$, $\partial h/\partial y$, and

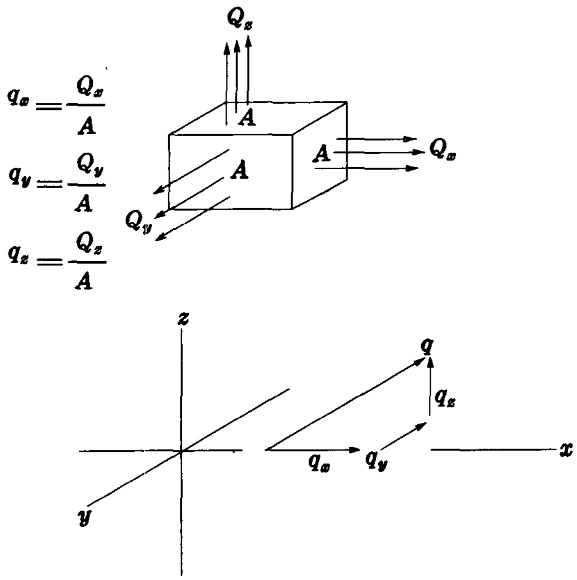


fig. *iv*

$\partial h/\partial z$—in addition to the time derivative. In such a case, as noted in the discussion at the close of Part II, we would utilize Darcy's law in a somewhat more general form. When flow may occur in more than one direction, we consider the specific discharge, $q=Q/A$ to be a vector, having the three components q_x, q_y, and q_z. If the medium is isotropic, each of these components is given by a form of Darcy's law, in which the partial derivative of head in the direction concerned is employed. The expressions for the apparent velocity components are

$$q_x = -K\frac{\partial h}{\partial x}$$

$$q_y = -K\frac{\partial h}{\partial y}$$

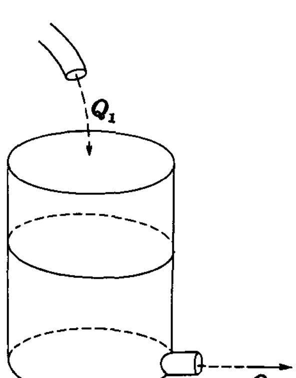
$$q_z = -K\frac{\partial h}{\partial z}$$

where K is the hydraulic conductivity.

q_x actually represents the fluid discharge per unit area in the x direction—that is, the discharge crossing a unit area oriented at right angles to the x axis. Similarly, q_y and q_z represent the discharges crossing unit areas normal to the y and z axes, respectively. The three components are calculated individually and added vectorially to obtain the resultant apparent velocity of the flow. (See figure *iv*.)

We now proceed to the programed material of Part V.

1



The picture shows an open tank with an inflow at the top and an outlet pipe at the base. Water is flowing in at the top at a rate Q_1 and is flowing out at the base at a rate Q_2.

QUESTION

Suppose we observe that the volume of water in the tank is increasing at a rate of 5 cubic feet per minute. Which of the following equations could we consider correct?

	Turn to Section:
$Q_1=5$ cubic feet per minute	29
$\frac{Q_1+Q_2}{2}=2.5$ cubic feet per minute	17
$Q_1-Q_2=5$ cubic feet per minute	21

"2

Your answer in Section 32,

$$Q_1 - Q_2 = K\frac{\partial h}{\partial x},$$

is not correct. The inflow through face 1 of the prism is given, according to Darcy's law, as a product of the hydraulic conductivity, the head gradient at face 1, and the cross-sectional area, $b\Delta y$, of face 1; that is,

$$Q_1 = -K\left(\frac{\partial h}{\partial x}\right)_1 b\Delta y.$$

Similarly, the outflow through face 2 is given as a product of hydraulic conductivity, head gradient at face 2, and the cross-sectional area of face 2, which is again $b\Delta y$; that is

$$Q_2 = -K\left(\frac{\partial h}{\partial x}\right)_2 b\Delta y.$$

Inflow minus outflow is thus given by

$$Q_1 - Q_2 = Kb\Delta y\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}.$$

In the preceeding sections, we have seen that the term

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}$$

can be written in an equivalent form using the second derivative.

Return to Section 32 and use this second derivative form in the above equation to obtain the correct answer.

"3

Your answer in Section 30,

$$Q_1 = \frac{-K}{b\Delta y}\left(\frac{\partial h}{\partial x}\right)_1,$$

is not correct. Darcy's law states that the flow through a given plane—in this case, face 1 of the prism—is given as the product of hydraulic conductivity, area, and head gradient. Your answer gives the flow as the product of hydraulic conductivity and head gradient, divided by area.

Return to Section 30 and choose another answer.

"4

Your answer in Section 7,

$$\frac{\partial^2 h}{\partial x^2}\cdot\frac{\partial h}{\partial x},$$

is not correct. We wish to find the change in the quantity $\partial h/\partial x$ over a small interval, Δx, of the x-axis. We have seen in the preceding sections of Part V that the change in a variable over such an interval is given by the derivative of the variable times the length of the interval. Here, the variable is $\partial h/\partial x$ and the interval is Δx; thus we require the derivative of $\partial h/\partial x$ with respect to x and must multiply this by the interval Δx.

Return to Section 7 and choose another answer.

5″

Your answer in Section 21 is not correct. A falling water level in the piezometer would indicate that water was being released from storage in the prism of aquifer. The slope of a plot of piezometer level versus time would in this case be negative; that is, $\partial h/\partial t$ would be negative, since h would decrease as t increased. According to the storage equation,

$$\frac{dV}{dt}=SA\frac{\partial h}{\partial t}$$

and therefore the rate of accumulation in storage, dV/dt, would also have to be negative. That is, we would have depletion from storage, rather than accumulation in storage. The question in Section 22, however, states that inflow to the prism exceeds outflow; thus, according to the equation of continuity, accumulation in storage should be occurring.

Return to Section 21 and choose another answer.

6″

Your answer in Section 21 is not correct. If the water level in the piezometer were constant with time, a plot of the piezometer readings versus time would simply be a horizontal line. The slope of such a plot, $\partial h/\partial t$, would be zero. From the storage equation, then, the rate of accumulation of water in storage in the prism would have to be zero, for we would have

$$\frac{dV}{dt}=SA\frac{\partial h}{\partial t}=SA\cdot 0=0.$$

The question states, however, that inflow to the prism exceeds outflow; according to the equation of continuity, then, the rate of accumulation of water in storage cannot be zero. Rather, it must equal the difference between inflow and outflow.

Return to Section 21 and choose another answer.

7″

Your answer in Section 16,

$$\left(\frac{dy}{dx}\right)_2-\left(\frac{dy}{dx}\right)_1=\left(\frac{d\left(\frac{dy}{dx}\right)}{dx}\right)_{1-2}(x_2-x_1),$$

is correct. In this case, the derivative itself is the variable whose change is required, and for this we must use the derivative of the derivative,

$$\frac{d\left(\frac{dy}{dx}\right)}{dx},$$

evaluated at an appropriate point within the interval. This term is called the second derivative of y with respect to x, and the notation d^2y/dx^2 is used for it. That is,

$$\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{dx}\right)}{dx}$$

$$=\text{slope of a plot of }\frac{dy}{dx}\text{ versus }x.$$

The terms and notations used in the case of partial derivatives are entirely parallel. The notation $\partial^2h/\partial x^2$ is used to represent the second *partial* derivative of h with respect to x, which in turn is simply the partial derivative of $\partial h/\partial x$ with respect to x. That is,

Con.— " 7

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial\left(\dfrac{\partial h}{\partial x}\right)}{\partial x}$$

$$= \text{slope of a plot of } \frac{\partial h}{\partial x} \text{ versus } x.$$

Again, the partial derivative notation indicates that we can expect $\partial h/\partial x$ to vary with t (or some other variable) as well as with x; $\partial^2 h/\partial x^2$ measures only its change due to a change in x, all other independent variables being held constant.

QUESTION

In Section 9, we saw that inflow minus outflow for our prism of aquifer could be expressed in the form

$$Q_1 - Q_2 = Kb\Delta y\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}$$

and that the term

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}$$

represented the change in the hydraulic gradient occurring across the prism. If the width of the prism in the x direction (that is, parallel to the x-axis) is Δx, which of the following expressions could most reasonably be substituted for

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}?$$

	Turn to Section:
$\dfrac{\partial^2 h}{\partial x^2} \cdot \dfrac{\partial h}{\partial x}$	4
$\dfrac{\partial\left(\dfrac{\partial h}{\partial x}\right)}{\partial x}$	23
$\dfrac{\partial^2 h}{\partial x^2} \cdot \Delta x$	32

" 8

Your answer in Section 30,

$$Q_1 = -Kb\Delta x\Delta y\left(\frac{\partial h}{\partial x}\right)_1,$$

is not correct. According to Darcy's law, the flow through face 1 should equal the product of the hydraulic conductivity, the cross-sectional area of the face, and the head gradient at face 1. The cross-sectional area of face 1 is simply $b\Delta y$.

Return to Section 30 and choose another answer.

" 9

Your answer in Section 33,

$$Q_1 - Q_2 = -Kb\Delta y\left\{\left(\frac{\partial h}{\partial x}\right)_1 - \left(\frac{\partial h}{\partial x}\right)_2\right\},$$

is correct. We may change the term in braces to $(\partial h/\partial x)_2 - (\partial h/\partial x)_1$ and drop the negative sign to obtain the form

$$Q_1 - Q_2 = Kb\Delta y\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\}.$$

The term $(\partial h/\partial x)_2 - (\partial h/\partial x)_1$ represents the change in hydraulic gradient from one side of the prism of aquifer to the other. We wish now to express this change in hydraulic gradient in a slightly different form.

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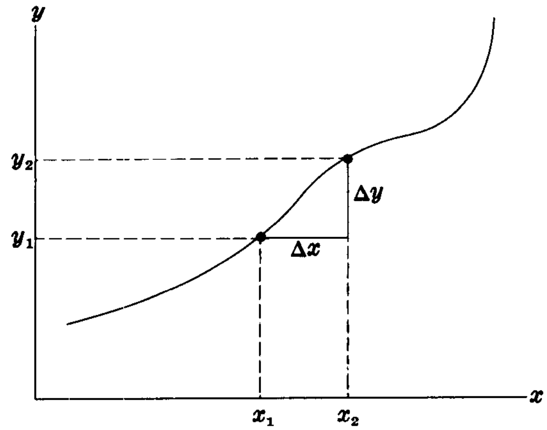
9″ —Con.



QUESTION

In the figure, a variable y is plotted as a function of an independent variable, x. As x changes from x_1 to x_2, y changes from y_1 to y_2; $(dy/dx)_{1-2}$ represents the slope of the plot at a point between x_1 and x_2. If the change in x is small, which of the following expressions would you use to obtain an approximate value for the change in y?

Turn to Section:

$$y_2-y_1=\left(\frac{dy}{dx}\right)_{1-2}(x_2-x_1) \qquad 16$$

$$y_2-y_1=\left(\frac{dy}{dx}\right)_{1-2}+(x_2-x_1) \qquad 25$$

$$y_2-y_1=m(x_2-x_1)+\frac{\Delta y}{\Delta x} \qquad 20$$

10″

Your answer in Section 34,

$$\frac{dV}{dt}=S\Delta x\Delta y\frac{\partial h}{\partial t}$$

is correct. (We should note that for a finite prism, $\partial h/\partial t$ may vary from point to point between the two faces; and we require an average value, which will yield the correct value of dV/dt for the prism. In fact there is always at least one point within the prism at which the value of $\partial h/\partial t$ is such an average, and we assume that we can measure and use $\partial h/\partial t$ at such a point. If we allow the prism to become infinitesimal in size, only one value of $\partial h/\partial t$ can be specified within it, and this value will yield an exact result for dV/dt.)

Using the equation of continuity we may now set this expression which we have obtained for rate of accumulation equal to our expression for inflow minus outflow.

QUESTION

Which of the following equations is obtained by equating the above expression for dV/dt to that obtained in Section 34 for Q_1-Q_2?

Turn to Section:

$$\frac{\partial^2 h}{\partial x^2}=\frac{S}{T}\frac{\partial h}{\partial t} \qquad 19$$

$$T\frac{\partial^2 h}{\partial x^2}\Delta x\Delta y=S\frac{\partial h}{\partial t} \qquad 11$$

$$T\Delta y\Delta x\frac{\partial h}{\partial x}=S\Delta x\Delta y\frac{\partial h}{\partial t} \qquad 24$$

11

Your answer in Section 10 is not correct. We used Darcy's law to obtain expressions for inflow and outflow from the prism of aquifer, and we used the second derivative notation to express the difference between inflow and outflow. This led, in Section 34, to the equation

$$Q_1 - Q_2 = T\Delta x\Delta y\frac{\partial^2 h}{\partial x^2}$$

for inflow minus outflow. According to the equation of continuity, inflow minus outflow must equal rate of accumulation in storage; that is

$$Q_1 - Q_2 = \frac{dV}{dt}.$$

We obtained an expression for dV/dt through the storage equation, which states that rate of accumulation in storage must equal the product of storage coefficient, surface (or base) area, and time rate of change of head; that is

$$\frac{dV}{dt} = S\Delta x\Delta y\frac{\partial h}{\partial t}.$$

Substitution of the first and third equations into the second will yield the correct result.

Return to Section 10 and choose another answer.

12

Your answer in Section 34,

$$\frac{dV}{dt} = \frac{S}{K}\frac{\partial h}{\partial t},$$

is not correct. The storage equation tells us that the rate of accumulation of water in storage within the prism of aquifer must equal the product of storage coefficient, rate of change of head with time, and base area of the prism. Hydraulic conductivity, K, is not involved in the storage equation. In the answer which you selected, there is no term describing the base area of the prism, and hydraulic conductivity appears on the right side of the equation.

Return to Section 34 and choose another answer.

13

Your answer in Section 16,

$$\left(\frac{dy}{dx}\right)_2 - \left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)^2_{1-2}(x_2 - x_1),$$

is not correct. In this case, the dependent variable, plotted on the vertical axis, is dy/dx. As we have seen in preceding sections, the change in the dependent variable is given by the slope of the graph, or derivative of the dependent variable with respect to x, multiplied by the change in x. Thus we require the derivative of dy/dx with respect to x in our answer. In the answer shown above, however, we have only the square of the derivative of y with respect to x.

Return to Section 16 and choose another answer.

14

Your answer in Section 22 is not correct. It is true that if inflow differs from outflow the water level in the prism of aquifer must change with time. However, it need not rise; if inflow is less than outflow, it will fall.

Return to Section 22 and choose another answer.

15

Your answer in Section 33,

$$Q_1-Q_2=\frac{S}{K}\left(\frac{\partial h}{\partial x}\right)_2$$

is not correct. This answer associates storage coefficient, S, with a space derivative of head, $(\partial h/\partial x)_2$; this in itself should be sufficient to indicate that it is incorrect. In the storage equation, S is associated with the time derivative of head, $\partial h/\partial t$. Again, the answer chosen involves only the head gradient at the outflow face. Since we are seeking an expression for inflow *minus* outflow, we would expect head gradients at both faces to be involved in the answer.

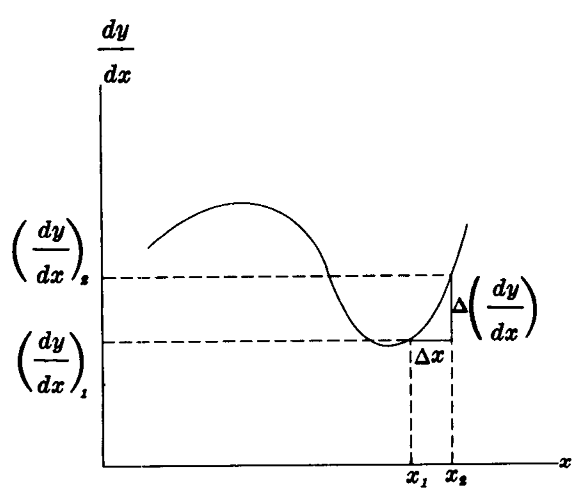
Return to Section 33 and choose another answer.

16

Your answer in Section 9,

$$y_2-y_1=\left(\frac{dy}{dx}\right)_{1-2}(x_2-x_1),$$

is correct. The change in the dependent variable, y, is found by multiplying the change in the independent variable, x, by the slope of the plot, dy/dx. Note that dy/dx must be the slope in the vicinity of the interval x_1 to x_2;



frequently, it is considered to be the slope at the midpoint of this interval. The approximation becomes more and more accurate as the size of the interval, x_2-x_1, decreases. The above equation is often written in the form

$$\Delta y=\frac{dy}{dx}\cdot\Delta x.$$

(In a more formal sense, it can be demonstrated that if y is a continuous function of x and if dy/dx exists throughout the interval from x_1 to x_2, then there is at least one point somewhere in this interval at which the derivative, dy/dx, has a value such that

$$\frac{dy}{dx}=\frac{y_2-y_1}{x_2-x_1}$$

or

$$y_2-y_1=\frac{dy}{dx}(x_2-x_1).$$

This is known as the law of the mean of differential calculus. It guarantees that the approximation can always be used, provided we are careful about the point within the interval at which we take dy/dx. Further, since this law must hold no mater how small

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the interval $(x_2 - x_1)$ is taken, the approximation must become exact as the interval is allowed to become infinitesimal.)

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QUESTION

Now suppose we measure the slope of our curve, dy/dx, at various points, and construct a plot of dy/dx versus x, as shown in the figure. Again, suppose we wish to know the change in dy/dx which occurs as x changes from x_1 to x_2. The subscript $1-2$ is again used to denote evaluation at a point between x_1 and x_2. Which of the following expressions would give an approximate value for this change?

	Turn to Section:
$\left(\frac{dy}{dx}\right)_2 - \left(\frac{dy}{dx}\right)_1 = (x_2 - x_1)\left(\frac{dy}{dx}\right)_{1-2}$	31
$\left(\frac{dy}{dx}\right)_2 - \left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)^2_{1-2}(x_2 - x_1)$	13
$\left(\frac{dy}{dx}\right)_2 - \left(\frac{dy}{dx}\right)_1 = \left(\frac{d\left(\frac{dy}{dx}\right)}{dx}\right)_{1-2}(x_2 - x_1)$	7

17

Your answer in Section 1 is not correct. The rate of accumulation in the tank does depend upon both Q_1 and Q_2, but not in the way that your answer implies. The inflow to the tank must be balanced by outflow, by accumulation of water in the tank, or by a combination of these factors.

Return to Section 1 and choose another answer.

18

Your answer in Section 33,

$$Q_1 - Q_2 = K\left(\frac{\partial h}{\partial x}\right)_1 - K\left(\frac{\partial h}{\partial x}\right)_2$$

is not correct. The answer treats both inflow and outflow as products of hydraulic conductivity and head gradient; but we have seen, in our application of Darcy's law to the problem, that each should be a product of hydraulic conductivity, head gradient, and flow area.

Return to Section 33 and choose another answer.

19

Your answer in Section 10,

$$\frac{\partial^2 h}{\partial x^2} = \frac{S}{T}\frac{\partial h}{\partial t},$$

is correct. This equation describes groundwater movement under the simple conditions which we have assumed—that is, where the aquifer is confined, horizontal, homogeneous, and isotropic, and the movement is in one direction (taken here as the x direction).[1] If horizontal components of motion normal to the x-axis were present, we would have to consider inflow and outflow through the other two faces of the prism; that is, the two faces normal to the y-axis. We would find this inflow minus outflow to be

$$Q_{y1} - Q_{y2} = Kb\Delta x\Delta y\frac{\partial^2 h}{\partial y^2}.$$

The total inflow minus outflow for the prism would then be $(Q_{x_1} - Q_{x_2}) + (Q_{y_1} - Q_{y_2})$,

[1] A rigorous and more general development of the ground water equation is given by Cooper (1966).

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where $Q_{x_1}-Q_{x_2}$ represents the term we obtained previously, $Kb\Delta x\Delta y(\partial^2 h/\partial x^2)$. Finally, equating this total inflow minus outflow to the rate of accumulation, we would have

$$Kb\Delta x\Delta y\frac{\partial^2 h}{\partial x^2}+Kb\Delta x\Delta y\frac{\partial^2 h}{\partial y^2}=S\Delta x\Delta y\frac{\partial h}{\partial t}$$

or, using the notation $T=Kb$, and dividing through by $T\Delta x\Delta y$,

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

These equations are *partial differential equations;* that is, they are equations containing partial derivatives. The relation given above for two-dimensional flow is a partial differential equation in three independent variables x, y, and t. For simplicity, we continue the discussion in terms of the equation for unidirectional flow,

$$\frac{\partial^2 h}{\partial x^2}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

This is a partial differential equation in two independent variables, x and t. It relates the rate of change of head with time, to the rate at which the slope of the potentiometric surface, $\partial h/\partial x$, changes with distance. When we say that we require a *solution* to this partial differential equation, we mean that we are looking for an expression giving head, h, as a function of position, x, and time, t, such that when this expression is differentiated twice with respect to x (to obtain $\partial^2 h/\partial x^2$) and once with respect to t (to obtain $\partial h/\partial t$), the results will satisfy the condition

$$\frac{\partial^2 h}{\partial x^2}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

As with ordinary differential equations, there will always be an infinite number of expressions which will satisfy a partial differential equation; the particular solution required for a given problem must satisfy, in addition, certain conditions peculiar to that problem. As in ordinary differential equations, these additional conditions, termed boundary conditions, establish the starting points from which the changes in h described by the differential equation are measured.

This concludes Part V. In Part VI, we will make a development similar to the one made in Part V, but using polar coordinates, and dealing with the problem of nonequilibrium flow to a well. Our approach will be the same: we will express inflow and outflow in terms of Darcy's law and rate of accumulation in terms of the storage equation; we will then relate these flow and storage terms through the equation of continuity. We will go on to discuss a particular solution of the resulting partial differential equation and will show how this solution can be used to build up other solutions, including the well-known Theis equation.

20″

Your answer in Section 9,

$$y_2-y_1=m(x_2-x_1)+\frac{\Delta y}{\Delta x},$$

is not correct. If y is plotted as a function of x, the change in y corresponding to a small change in x is given by the relation

Change in $y=$ (Slope of curve) $\cdot$ (Change in x),

where the slope of the curve is measured in the vicinity in which the change is sought. This follows directly from the definition of the slope of the curve.

Return to Section 9 and choose another answer.

21

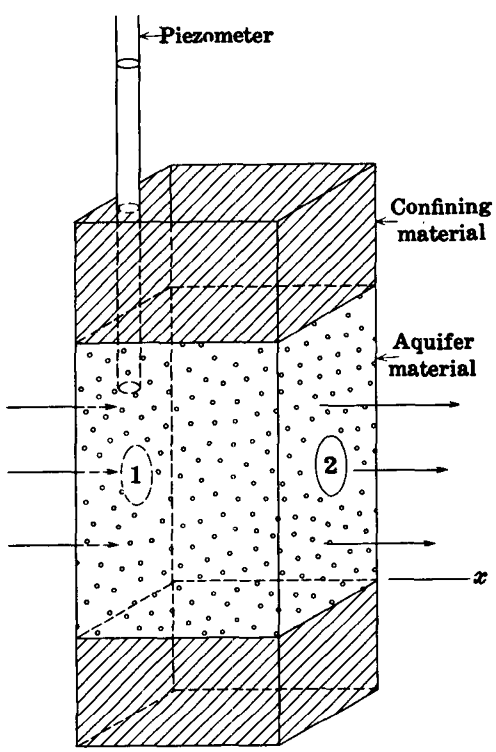
Your answer in Section 1 is correct. If water is accumulating in the tank at a rate of 5 cubic feet per minute, inflow must exceed outflow by this amount. This is essentially a statement of the principle of conservation of mass. Since matter cannot be destroyed (except by conversion into energy, which we need not consider here), the difference between the rate at which mass enters the tank and that at which it leaves the tank must equal the rate at which it accumulates in the tank. Further, because compression of the water is not significant here, we may use volume in place of mass. In general terms, the relation with which we are dealing may be stated as:

Inflow − Outflow = Rate of accumulation.

This relation is often termed the equation of continuity.

Note that if outflow exceeds inflow, the rate of accumulation will be negative—that is, we will have depletion rather than accumulation. An important special case of this equation is that in which inflow and outflow are in balance, so that the rate of accumulation is zero. As an example, consider a tank in which the inflow is just equal to the outflow. Rate of accumulation in the tank is zero, and the water level does not change with time. The flow is said to be in equilibrium, or in the steady state. The problems which we considered in Part III were of this sort; no changes of head with time were postulated, so the assumption that inflow and outflow were in balance was implicit. The flow pattern could be expected to remain the same from one moment to the next.

Forms of the equation of continuity occur in all branches of physics. In electricity, for example, if the flow of charge toward a capacitor exceeds that away from it, charge must accumulate on the capacitor plate, and voltage must increase. In heat conduction, if the flow of heat into a region exceeds that leaving it, heat must accumulate within the region, and the temperature within the region must rise.

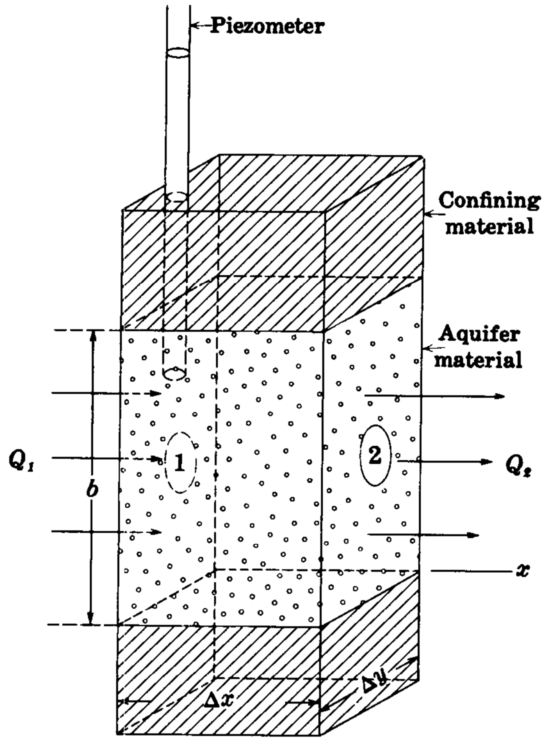


QUESTION

The sketch shows a prismatic section through a confined aquifer. Water is flowing in the x direction, that is, into the prism through face 1 and out of the prism through face 2. A piezometer or observation well measures the hydraulic head within the prism. Let us suppose that the volumetric rate at which water is entering through face 1 exceeds that at which it is leaving through face 2. The water level in the piezometer will then:

	Turn to Section:
remain constant with time	6
fall steadily	5
rise	30

22



Your answer in Section 30,

$$Q_1 = -Kb\Delta y\left(\frac{\partial h}{\partial x}\right)_1,$$

is correct. $(\partial h/\partial x)_1$ is the hydraulic gradient at the particular point and time in which we are interested. We simply insert it in Darcy's law to obtain the required flow rate.

We are dealing with nonequilibrium flow here; that is, in general, inflow and outflow will not be equal. Flow occurs only in the x direction; thus the outflow from our prism of aquifer must take place entirely through face 2, as shown in the sketch.

QUESTION

Assuming that outflow differs from inflow and that the hydraulic conductivity and thickness of the aquifer are constant, which of the following statements is correct?

	Turn to Section:
The water level in the prism must rise	14
The hydraulic gradient at face 2 of the prism must differ from that at face 1 of the prism	33
The rate of withdrawal from storage must be given by Darcy's law.	26

23

Your answer in Section 7,

$$\frac{\partial\left(\frac{\partial h}{\partial x}\right)}{\partial x},$$

is not correct. As we have seen in earlier sections of this chapter, the change in a dependent variable, over a small interval of the x-axis, Δx, is given by the derivative of the variable times the length of the interval. Here, the variable is $\partial h/\partial x$ and the term $\partial(\partial h/\partial x)/\partial x$ of your answer is certainly its derivative. However, this derivative is not multiplied by the interval along the x-axis; thus the answer gives only the *rate* of change of $\partial h/\partial x$ with distance—not its actual change across the interval Δx.

Return to Section 7 and choose another answer.

24

Your answer in Section 10 is not correct. The rate of accumulation in storage is given by

$$S\Delta x\Delta y\frac{\partial h}{\partial t},$$

as in the answer which you chose. However, the expression for inflow minus outflow requires a second derivative, as it deals with

Con.— " 24

the difference between two flow terms, each of which incorporates a first derivative. In the answer which you chose, inflow minus outflow is expressed in terms of a first derivative.

Review Sections 9, 32, and 34 and then return to Section 10 and choose another answer.

" 25

Your answer in Section 9,

$$y_2-y_1=\left(\frac{dy}{dx}\right)_{1-2}+(x_2-x_1),$$

is not correct. From the definition of slope, the change in y can be found by multiplying the change in x by the slope of the curve, measured in the interval x_1 to x_2. In the answer which you chose, the slope of the curve is added to the change in x.

Return to Section 9 and choose another answer.

" 26

Your answer in Section 22 is not correct. Darcy's law describes the transmission of ground water, not its withdrawal from storage. The storage equation, developed in Part IV, deals with changes in the quantity of water in storage.

Return to Section 22 and choose another answer.

" 27

Your answer in Section 32,

$$Q_1-Q_2=K\frac{\partial^2 h}{\partial x^2}\Delta x,$$

is not correct. Your answer includes the hydraulic conductivity, K, and the term

$$\frac{\partial^2 h}{\partial x^2}\Delta x,$$

which, as we have seen, is equal to

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2-\left(\frac{\partial h}{\partial x}\right)_1\right\}.$$

Thus if we were to expand your answer, expressing it in the original head gradient terms, we would have

$$Q_1-Q_2=K\left\{\left(\frac{\partial h}{\partial x}\right)_2-\left(\frac{\partial h}{\partial x}\right)_1\right\}=K\left(\frac{\partial h}{\partial x}\right)_2-K\left(\frac{\partial h}{\partial x}\right)_1.$$

This states that inflow is a product of hydraulic conductivity and head gradient, and that outflow is similarly a product of hydraulic conductivity and head gradient. We know from Darcy's law, however, that both inflow and outflow must be given as products of hydraulic conductivity, head gradient, and flow area. Your answer thus fails to incorporate flow area into the expression for inflow minus outflow.

Return to Section 32 and choose another answer.

28

Your answer in Section 34,

$$\frac{dV}{dt} = Sb\Delta x \frac{\partial h}{\partial t},$$

is not correct. The storage equation states that the rate of accumulation of water in storage in the prism of aquifer is equal to the product of storage coefficient, rate of change of head with time, and *base* area of the prism. In your answer the rate of accumulation is equated to the product of the storage coefficient, the rate of change of head with time, and the area, $b\Delta x$, of one of the vertical faces of the prism.

Return to Section 34 and choose another answer.

29

Your answer in Section 1 is not correct. Some of the inflow to the tank is balanced by outflow at the base. In order for your answer to be correct, the outflow, Q_2, would have to be zero. Only in that case would the rate of accumulation in the tank equal the inflow.

Return to Section 1 and choose another answer.

30

Your answer in Section 21 is correct. According to the equation of continuity, if inflow to the prism of aquifer exceeds outflow, water must be accumulating in storage within the prism. According to the storage equation, if water is accumulating in storage within the prism, hydraulic head in the prism must be increasing with time. Specifically, we have

$$\text{Inflow} - \text{Outflow} = \text{Rate of accumulation,}^{1}\ dV/dt$$

and

$$\frac{dV}{dt} = SA\frac{\partial h}{\partial t}$$

where A is the base area of the prism. Therefore,

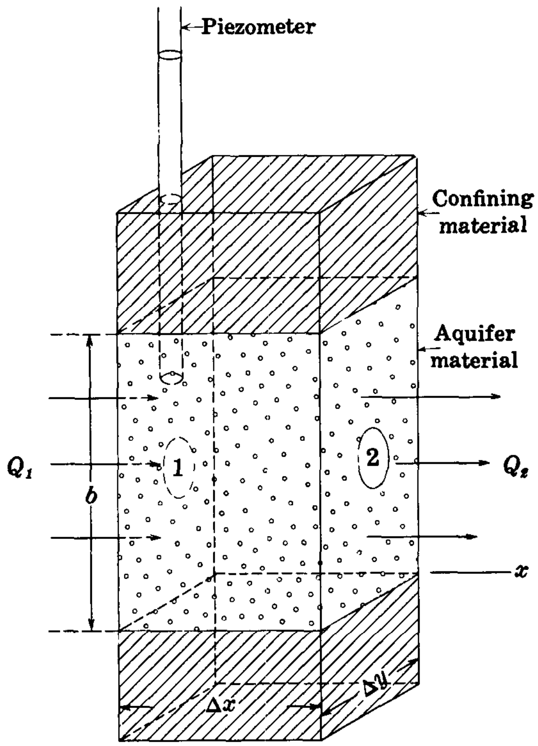
$$\text{Inflow} - \text{Outflow} = SA\frac{\partial h}{\partial t}.$$

If the term (Inflow-Outflow) is positive —that is, if inflow exceeds outflow—then $\partial h/\partial t$ must be positive, and water levels must be increasing with time. In the above equations, we have used the partial derivative of head with respect to time, $\partial h/\partial t$; and in the equations that follow, we will use the partial derivative of head with respect to distance, $\partial h/\partial x$. These notations are used because, in this problem, head will vary both with time and with distance.

QUESTION

The sketch again shows the prism of Section 21. We assume this prism to be taken in a homogeneous and isotropic aquifer which is horizontal and of uniform thickness. Suppose we let $(\partial h/\partial x)_1$ represent the hydraulic gradient (in the x direction, which is the direction of the flow) at face 1 of the prism. We wish to write an expression for the inflow through face 1 of the prism. Let

[1] Here again we use volume in place of mass in the equation of continuity, even though slight compression and expansion of the water can be a factor contributing to confined storage. The changes in fluid density from point to point in a normal groundwater situation are sufficiently small to permit this approximation. In fact, if this were not the case, it would not be possible to use the simple formulation of storage coefficient, defined in terms of fluid volume, which we have adopted.



Con.— " **30**

us denote this inflow Q_1, and let us further denote the height of the prism (thickness of the aquifer) by b. The width of the prism normal to the x axis is denoted Δy, the length of the prism along the x axis is denoted Δx, and the hydraulic conductivity of the aquifer is denoted K. Which of the following equations gives the required expression for the inflow at face 1?

	Turn to Section:
$Q_1 = -Kb\Delta y\left(\dfrac{\partial h}{\partial x}\right)_1$	22
$Q_1 = -Kb\Delta x\Delta y\left(\dfrac{\partial h}{\partial x}\right)_1$	8
$Q_1 = \dfrac{-K}{b\Delta y}\left(\dfrac{\partial h}{\partial x}\right)_1$	3

" **31**

Your answer in Section 16,

$$\left(\frac{dy}{dx}\right)_2 - \left(\frac{dy}{dx}\right)_1 = (x_2 - x_1)\left(\frac{dy}{dx}\right)_{1-2},$$

is not correct. In the preceding sections we saw that the change in the dependent variable is given by the change, $x_2 - x_1$, in the independent variable, times the derivative of the dependent variable with respect to x. Here the dependent variable is dy/dx; but in your answer we do not have the derivative of this dependent variable with respect to x —we have, rather, only the derivative of y with respect to x.

Return to Section 16 and choose another answer.

" **32**

Your answer in Section 7,

$$\frac{\partial^2 h}{\partial x^2} \cdot \Delta x,$$

is correct. This term is equivalent to the term

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2 - \left(\frac{\partial h}{\partial x}\right)_1\right\},$$

provided that we choose a suitable point within the interval $x_2 - x_1$ at which to evaluate $\partial^2 h/\partial x^2$. The product $(\partial^2 h/\partial x^2)\ \Delta x$ represents the slope of a plot of $\partial h/\partial x$ versus x, multiplied by the interval along the x -axis, Δx, and thus gives the change in $\partial h/\partial x$ over this interval.

(*continued on next page*)

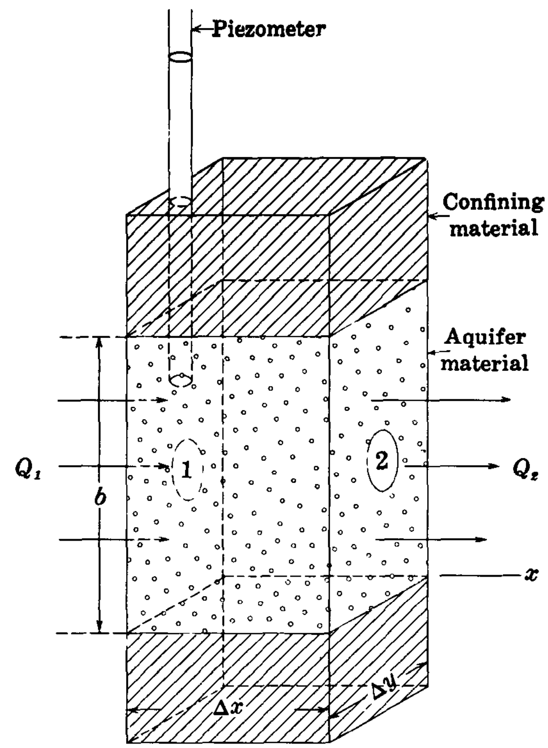
32″—Con.

QUESTION

Using this expression for

$$\left\{\left(\frac{\partial h}{\partial x}\right)_2-\left(\frac{\partial h}{\partial x}\right)_1\right\},$$

which of the following forms is the correct expression for inflow minus outflow, Q_1-Q_2, for our prism of aquifer, which is shown again in the diagram?



	Turn to Section:
$Q_1-Q_2=K\frac{\partial^2 h}{\partial x^2}\Delta x$	27
$Q_1-Q_2=Kb\Delta y\Delta x\frac{\partial^2 h}{\partial x^2}$	34
$Q_1-Q_2=K\frac{\partial h}{\partial x}$	2

33″

Your answer in Section 22 is correct. If we apply Darcy's law at face 2, we have

$$Q_2=-Kb\Delta y\left(\frac{\partial h}{\partial x}\right)_2$$

where at face 1 we had

$$Q_1=-Kb\Delta y\left(\frac{\partial h}{\partial x}\right)_1.$$

K, b, and Δy do not change. Thus if the outflow, Q_2, is to differ from the inflow, Q_1, the hydraulic gradients at the inflow and outflow faces must differ—that is, $(\partial h/\partial x)_2$ must differ from $(\partial h/\partial x)_1$.

QUESTION

Using the expressions we have developed for inflow and outflow, which of the following terms would describe inflow *minus* outflow for the prism?

	Turn to Section:
$Q_1-Q_2=K\left(\frac{\partial h}{\partial x}\right)_1-K\left(\frac{\partial h}{\partial x}\right)_2$	18
$Q_1-Q_2=\frac{S}{K}\left(\frac{\partial h}{\partial x}\right)_2$	15
$Q_1-Q_2=-Kb\Delta y\left\{\left(\frac{\partial h}{\partial x}\right)_1-\left(\frac{\partial h}{\partial x}\right)_2\right\}$	9

34

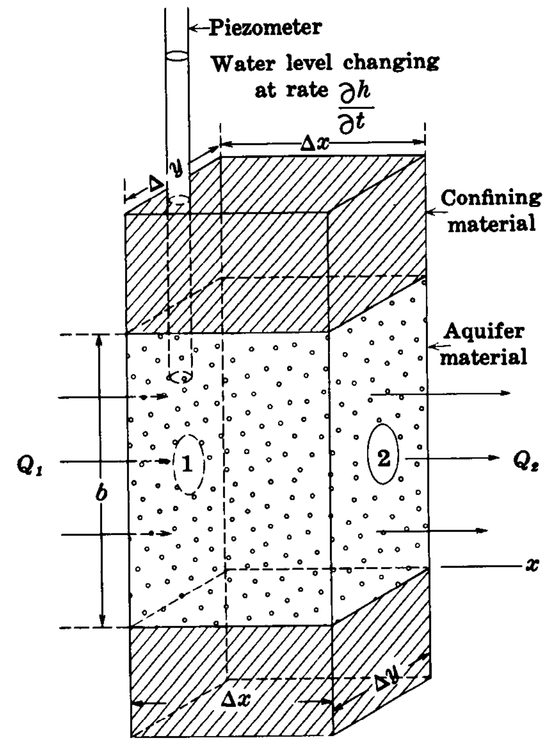
Your answer in Section 32,

$$Q_1 - Q_2 = Kb\Delta y\Delta x\frac{\partial^2 h}{\partial x^2},$$

is correct. The term Kb, representing the hydraulic conductivity of the aquifer times its thickness, is called the transmissivity or transmissibility of the aquifer, and is designated by the letter T. Using this notation, the expression for inflow minus outflow becomes

$$Q_1 - Q_2 = T\Delta y\Delta x\frac{\partial^2 h}{\partial x^2}.$$

Now according to the equation of continuity, this inflow minus outflow must equal the rate of accumulation of water in storage within the prism of aquifer, which is shown in the figure.



QUESTION

We represent the average time rate of change of head in the prism of aquifer by $\partial h/\partial t$ and note that the base area of the prism is $A = \Delta x\Delta y$. Using the storage equation, which of the following expressions gives the rate of accumulation in storage within the prism?

	Turn to Section:
$\frac{dV}{dt} = Sb\Delta x\frac{\partial h}{\partial t}$	28
$\frac{dV}{dt} = \frac{S}{K}\frac{\partial h}{\partial t}$	12
$\frac{dV}{dt} = S\Delta x\Delta y\frac{\partial h}{\partial t}$	10