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Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B2

# INTRODUCTION TO GROUND-WATER HYDRAULICS

## A Programed Text for Self-Instruction

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Book 3

APPLICATIONS OF HYDRAULICS

# Part II. Darcy's Law

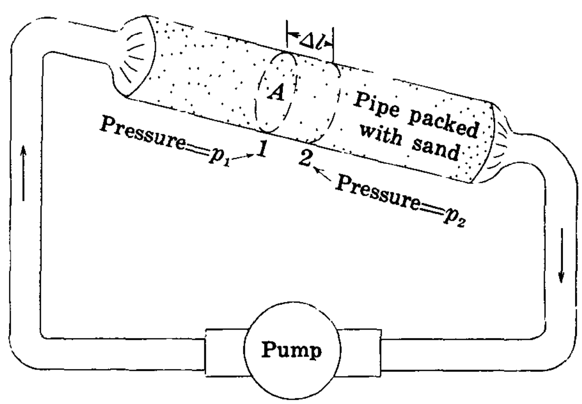
## Introduction

Part II gives a development of Darcy's law. This law relates specific discharge, or discharge per unit area, to the gradient of hydraulic head. It is the fundamental relation governing steady-state flow in porous media. The development given here should not be taken as a rigorous derivation; it is no more than a plausibility argument, and is presented in order to give the reader some appreciation for the physical significance of the relation.

Following the program section of Part II a short discussion on generalization of Darcy's law is given in text format.

## (1)

In mechanics, when considering the steady motion of a particle, it is customary to equate the forces producing the motion to the frictional forces opposing it. The same approach may be followed in considering the steady movement of fluid through a porous medium. In studying the motion of a solid particle through a fluid, we find that the force of friction opposing the motion is proportional to the velocity of the particle. Similarly, in flow through a porous medium, we will assume that the frictional forces opposing the flow are proportional to the fluid velocity. Our approach, then, will be to obtain expressions for the forces driving a flow and to equate these to the frictional force opposing the flow, which will be assumed proportional to the velocity. More exactly, we will take the vector sum of the forces driving and opposing the flow and set this equal to zero. What we are saying is that because the fluid motion is steady—that is, because no acceleration is observed—the forces on the fluid must be in balance, and therefore that their vector sum is zero, at all points. The equation that we obtain from this process of balancing forces will be a form of Darcy's law. We begin by considering the forces which drive the flow.




#### QUESTION

Suppose we have a pipe packed with sand, as in the diagram. The porosity of the sand is $n$. Liquid of density $\rho$ is circulated through the pipe by means of a pump. The dotted lines mark out a small cylindrical segment in the pipe, of length $\Delta l$, and of cross-sectional area $A$, equal to that of the pipe. A

## (1)—Con.

small volume, or element, of the moving fluid occupies this segment. The fluid pressure at point 1, at the upstream side of the segment, is $p_1$.

Which of the following expressions would best represent the force exerted on the upstream face of the fluid element by the adjacent fluid element?

| | Turn to Section: |
|---|---|
| $p_1A$ | 25 |
| $p_1nA$ | 8 |
| $p_1\rho g$ | 16 |

## (2)

Your answer in Section 19,

$$-\frac{1}{k}\mu Q(\Delta l\cdot n\cdot A),$$

is not correct. Our assumptions were that the frictional retarding force would be proportional in some way to the dynamic viscosity ($\mu$), to the volume of fluid in the element ($\Delta l\cdot n\cdot A$), and to the specific discharge, or flow per unit area ($Q/A$). While the answer which you have chosen is not incompatible with these assumptions, it does not fit them as well as one of the other answers. Your answer assumes the retarding force to be proportional more particularly to the full discharge, $Q$, than to the specific discharge, $Q/A$.

**Return to Section 19 and choose another answer.**

## (3)

Your answer in Section 26 is not correct. The term $\Delta l\cdot n\cdot A$ gives the volume of fluid in the element; the question asked for the mass of fluid in the element. Keep in mind that $\rho$, the density of the fluid, represents its mass per unit volume.

**Return to Section 26 and choose another answer.**

## (4)

Your answer in Section 35 is not correct. The term $\sqrt{(\Delta z)^2+(\Delta x)^2}$ is obviously equal to $\Delta l$, so that the answer you selected is equivalent to the term $\rho\cdot n\cdot A\cdot g\cdot\Delta l$. But as we saw in Section 15, this term gives the magnitude of the *total* gravitational force on our fluid element; what we want here is an expression for the *component* of this total force in the direction of flow. We have seen that this component is given by the expression $\rho\cdot n\cdot A\cdot g\cdot\Delta l\cdot\cos\gamma$; the idea of the question is to find a term equivalent to $\cos\gamma$ and to substitute it into the above expression.

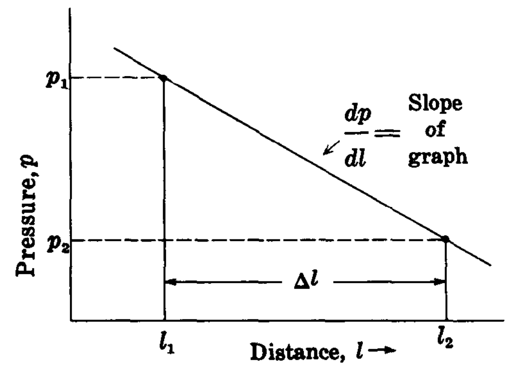
**Return to Section 35 and choose another answer.**

## (5)

Your answer in Section 31,

$$\frac{dp}{dl}nA,$$

is not correct. The expression obtained previously for the net force was $(p_1-p_2)nA$, or $-\Delta pnA$. You have substituted the pressure *gradient,* or rate of pressure change per foot, for the small pressure *change,* $-\Delta p$. To obtain a net change, or increment, from a gradient, or rate of change per unit distance, we must multiply the rate per unit distance by the distance over which this change takes place. For example, $dp/dl$ in the figure represents the slope of a graph of pressure, $p$, versus distance, $l$. To obtain the pressure *change*, $p_2-p_1$, we must multiply this slope by the length of the interval, $\Delta l$; and since we actually require the quantity $p_1-p_2$, we must insert a negative sign. (In the situation shown at left, $p_1$ is greater than $p_2$—that is, pressure is decreasing in the direction of flow, $l$. The derivative $dp/dl$ is therefore an intrinsically negative quantity itself—the graph has a negative slope. By inserting another negative sign, we will obtain a positive result for the term $p_1-p_2$.)

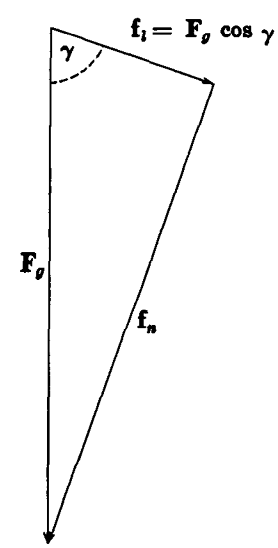



$p_2-p_1=$ Pressure change, $\Delta p=\frac{dp}{dl}\times\Delta l$

**Return to Section 31 and choose another answer.**

## (6)

Your answer in Section 33 is not correct. The term $\rho\cdot n\cdot\Delta l\cdot A\cdot g$ gives the magnitude of the total gravitational force vector, $\mathbf{F}_g$. However, we require the component of this force vector in the direction $l$ since only this component is effective in producing flow along the pipe. In the vector diagram, the length of the arrow representing the gravitational force, $\mathbf{F}_g$, is proportional to the magnitude of that force, and the length of the arrows representing the two components, $\mathbf{f}_l$ and $\mathbf{f}_n$, are proportional to the magnitudes of those components. Using a diagram to show the resolution of a vector into its components makes it easy to visualize the following general rule: the magnitude of the component of a vector in a given direction is obtained by multiplying the magnitude of the vector by the cosine of the angle between the direction of the vector and the direction in which the component is taken.




**Return to Section 33 and choose another answer.**

## (7)

Your answer in Section 28,

$$\frac{Q}{A} = -K\frac{dh}{dl},$$

is correct. This relation between specific discharge and head gradient, or hydraulic gradient, $dh/dl$, was obtained experimentally by Henri Darcy (1856) and is known as Darcy's law for flow through porous media. The constant $K$, in the current usage of the U.S. Geological Survey, is termed the hydraulic conductivity and has the dimensions of a velocity. The constant $k$, again in the current usage of the Geological Survey, is termed the intrinsic permeability; it's dimensions are $(\text{length})^2$, and its units depend upon the units of density and viscosity employed. In the current usage of the Geological Survey, where $\rho$ is measured in $kg/m^3$, $g$ in $m/s^2$, and $\mu$ in $kg/(m\,s)$, $k$ would have the units of $m^2$.

As noted in Section 28, hydraulic conductivity, $K$, is related to intrinsic permeability, $k$, by the equation

$$K = k\frac{\rho g}{\mu}$$

where $\rho$ is the fluid density, $\mu$ the dynamic viscosity of the fluid, and $g$ the gravitational constant. Hydraulic conductivity thus incorporates two properties of the fluid and cannot be considered a property of the porous medium alone. Intrinsic permeability, on the other hand, is normally considered to be only a property of the porous medium. In groundwater systems, variations in density are normally associated with variations in dissolved-mineral content of the water, while variations in viscosity are usually due to temperature changes. Thus in problems involving significant variations in mineral content or in water temperature, it is preferable to utilize intrinsic permeability.

The entire theory of steady-state flow through porous media depends upon Darcy's law. There are certain more general forms in which it may be expressed to deal with three-dimensional motion; some of these are considered in the text-format discussion at the end of this chapter. The development presented in this chapter involves numerous arbitrary assumptions, and thus should not be considered a theoretical derivation of Darcy's law. It has been presented here to illustrate, in a general way, the physical significance of the terms appearing in the law.

**QUESTION**

Consider the following statements:

(a) ground water flows from higher elevations to lower elevations.
(b) ground water flows in the direction of decreasing pressure.
(c) ground water moves in the direction of decreasing head.

Based on Darcy's law as given in this chapter, which of these statements should always be considered true?

| | Turn to Section: |
|---|---|
| all three | 29 |
| (b) and (c) but not (a) | 13 |
| only (c) | 21 |

## (8)

Your answer, $p_1 nA$, in Section 1 is correct. The overall cross-sectional area of the upstream face of the segment is $A$. The area of fluid in the upstream face is $nA$, if we assume the ratio between fluid area and overall area to be equal to the porosity. The pressure, or force per unit area, multiplied by the fluid area then gives the total force on the fluid element through the upstream face. Similarly, if $p_2$ is the fluid pressure at the downstream face, $p_2 nA$, gives the magnitude of the force exerted on the downstream face of the fluid element by the adjacent downstream element.

## (8)—Con.

QUESTION

Let us assume that the pressure $p_1$ is greater than the pressure $p_2$. Which of the following expressions would best represent the *net* pressure-force on the element in the direction of flow?

| | Turn to Section: |
|---|---|
| $p_1nA+p_2nA$ | 23 |
| $\frac{p_1nA+p_2nA}{2}$ | 12 |
| $p_1nA-p_2nA$ | 31 |

## (9)

Your answer in Section 28 is not correct. We saw in Part I that head, $h$, was given by

$$h=\frac{p}{\rho g}+z.$$

It follows that

$$\frac{dh}{dl}=\frac{d(p/\rho g)}{dl}+\frac{dz}{dl}.$$

**Use this result in selecting a new answer to the question of Section 28.**

## (10)

Your answer in Section 11 is not correct. We have obtained expressions for two forces acting in the direction of flow—the net pressure force, which was calculated as the difference between forces exerted on the upstream and downstream faces of the element by adjacent elements of fluid (see Section 26); and the component of the gravitational force in the direction of flow (see Section 11). The question asks for the combined net force due to both pressure and gravity. Forces are combined by means of vector addition. In this case, however, the net pressure force and the component of gravity we are considering are oriented in the same direction—in the direction of flow. Vector addition in this instance therefore becomes a simple addition of the magnitudes of the two terms.

**Return to Section 11 and choose another answer.**

## (11)

Your answer,

$$\rho\cdot n\cdot\Delta l\cdot A\cdot g\frac{\Delta z}{\Delta l},$$

in Section 35, is correct. $\Delta z/\Delta l$ is the equivalent of cos $\gamma$; it simply gives the change in elevation per unit distance *along the path of flow*. (It thus differs from *slope* which by definition is the change in elevation per unit horizontal distance.) In the notation of calculus, $\Delta z/\Delta l$ would be represented by the derivative, $dz/dl$, implying the limiting value of the ratio $\Delta z/\Delta l$ as smaller and smaller values of $\Delta l$ are taken. The force component along the pipe must be positive, or oriented in the direction of flow, if $z$ decreases in the direction of flow—that is, if $dz/dl$ is negative. It must be negative, or oriented against the flow, if $z$ increases in the direction of flow—that is if $dz/dl$ is positive. We therefore introduce a negative sign, so that we have finally

$$\mathbf{f}_l=-\rho\cdot n\cdot A\cdot\Delta l\cdot g\cdot dz/dl$$

where $\mathbf{f}_l$ is the component of the gravitational

## (11)—Con.

force parallel to the pipe, as in Section 33. The total force driving the flow is the sum of this gravity component and the pressure force.

QUESTION

Which of the following expressions would give the net force on the fluid in the direction of flow, due to pressure and gravity together?

| | Turn to Section: |
|---|---|
| $\left(-\frac{dp}{dl}-\rho g\frac{dz}{dl}\right)\Delta l\cdot n\cdot A$ | 19 |
| $-\frac{dp}{dl}\cos\gamma+\rho\cdot n\cdot\Delta l\cdot A\cdot g\cdot\frac{dz}{dl}$ | 24 |
| $-\rho\cdot n\cdot\Delta l\cdot A\cdot g\cdot\frac{dz}{dl}-\frac{dp}{dl}$ | 10 |

## (12)

Your answer in Section 8 is not correct. The expression $(p_1nA+p_2nA)/2$ would be approximately equal to the force in the direction of flow against a cross-sectional area taken at the midpoint of our fluid element; it does not give the *net* force on the element itself in the direction of flow.

The fluid element extends along the pipe a short distance. Over this distance, pressure decreases from $p_1$ at the upstream face to $p_2$ at the downstream face. The force on the element at the upstream face is the force acting in the direction of flow; the force on the element at the downstream face is a force acting agaisnt the direction of flow. That is, it is a "back push" from the adjacent fluid element, against the element we are considering. Its magnitude is again given as a product of pressure, porosity, and face area, $p_2nA$, but we now insert a negative sign to describe the fact that it acts in opposition to the force previously considered. The net force in the direction of flow is obtained by algebraic addition of the two force terms.

**Return to Section 8 and choose another answer.**

## (13)

Your answer in Section 7 is not correct. Ground water frequently percolates downward from the water table; the pressure is greater at depth than at the water table, so in these cases water is moving in the direction of increasing pressure. Keep in mind that Darcy's law relates flow per unit area to the gradient of head, not to the gradient of pressure.

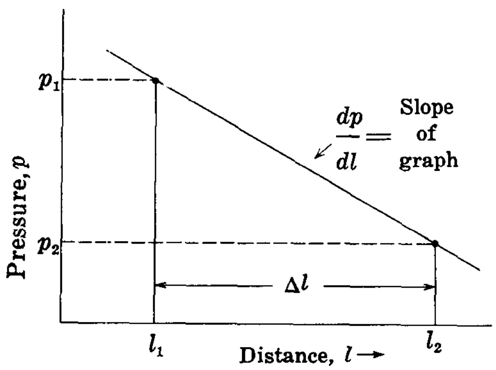
**Return to section 7 and choose another answer.**

## (14)

Your answer in Section 31 is not correct. We have seen that the net pressure force was equal to $-\Delta p n A$. It cannot be equal to this and to $\Delta p(dp/dl)nA$ (unless $dp/dl$ happens to equal $-1$, in a particular case).

We wish to substitute an expression involving the derivative, $dp/dl$, in place of the pressure change term, $-\Delta p$. To obtain an expression for a change, or an increment, from a derivative, it is necessary to multiply the derivative—that is, the rate of change per unit distance—by the distance over which the increment or change occurs. For example, the diagram shows a graph of pres-

## (14)—Con.



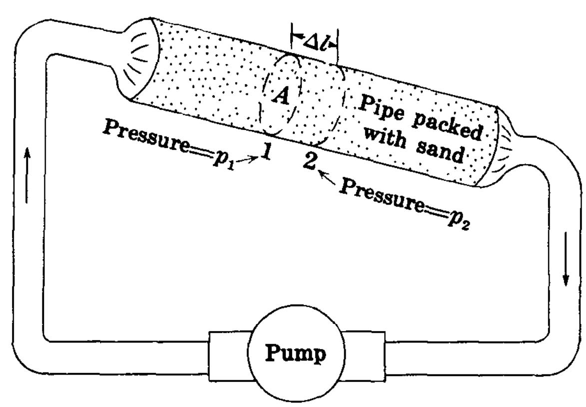

$$p_2 - p_1 = \text{Pressure change}, \Delta p = \frac{dp}{dl} \times \Delta l$$

sure versus distance. The slope of this graph is the derivative, $dp/dl$. If we wish to obtain the change in pressure, $p_2—p_1$ occurring over the interval $\Delta l$, we must multiply the rate of change per unit distance, $dp/dl$, by the distance $\Delta l$. Since we actually require the negative of this quantity, $p_1—p_2$, we must insert a negative sign. (As shown on the graph, $p_1$ exceeds $p_2$—pressure is decreasing in the direction of flow, $l$. The derivative of pressure with respect to distance, $dp/dl$, is therefore a negative quantity itself—that is, the graph has a negative slope. By inserting another negative sign, we will obtain a positive result for the term $p_1—p_2$.)

**Return to Section 31 and choose another answer.**

## (15)

Your answer, $m=\rho \cdot \Delta l \cdot n \cdot A$, in Section 26 is correct; mass density, $\rho$, times volume of fluid, $n \cdot \Delta l \cdot A$, where $n$ is porsity, gives the mass of fluid. The magnitude of the total force of gravity on our fluid element will, therefore, be $\rho \cdot \Delta l \cdot n \cdot A \cdot g$. This gravitational force acts vertically downward. As a force, however, it is a vector quantity; and like any other vector quantity it can be resolved into components acting in other directions.




**QUESTION**

The diagram again shows the flow system we have postulated. Which of the following statements is correct?

**Turn to Section:**

| | |
|---|---|
| The entire gravitational force is effecttive in causing flow along the pipe. | 22 |
| Only the component of the gravitational force parallel to the axis of the pipe contributes to flow along the pipe. | 33 |
| Only the horizontal component of the gravitational force contributes to flow along the pipe. | 18 |

## (16)

Your answer in Section 1 is not correct. The force on the element will be given by the pressure, or force per unit area, multiplied by the area of fluid against which the pressure acts.

**Return to Section 1 and choose another answer.**

## (17)

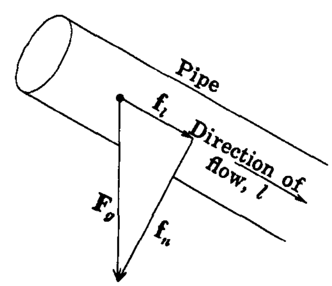
Your answer in Section 26 is not correct. The term $\rho \cdot \Delta l \cdot A$ would give the mass of a fluid element having a volume $\Delta l \cdot A$. In our problem, however, only a part of the volume $\Delta l \cdot A$ is occupied by fluid; the balance is occupied by solid sand grains, so that the actual volume of fluid is less than $\Delta l \cdot A$.

**Return to Section 26 and choose another answer.**

## (18)

Your answer in Section 15 is not correct. Gravity, as we are considering it, has no horizontal component. No vector can have a component perpendicular to its own direction. For our purposes we consider the gravitational force vector, $\mathbf{F}_g$, to be always directed vertically downward; there can be no horizontal component of this force.

The diagram shows the gravitational force vector resolved into two components—one parallel to the direction of flow, $\mathbf{f}_l$, and one perpendicular to the direction of flow, $\mathbf{f}_n$. Fluid velocity itself may be considered a vector, in the direction $l$. As such, it has no component in the direction of $\mathbf{f}_n$, normal to the pipe—and a force component normal to the pipe could not contribute in any way to the fluid velocity.




**Return to Section 15 and choose another answer.**

## (19)

Your answer in Section 11,

$$\left(-\frac{dp}{dl}-\rho g\frac{dz}{dl}\right)\Delta l \cdot n \cdot A$$

is correct. The net force *per unit volume of fluid* due to pressure and gravity would thus be

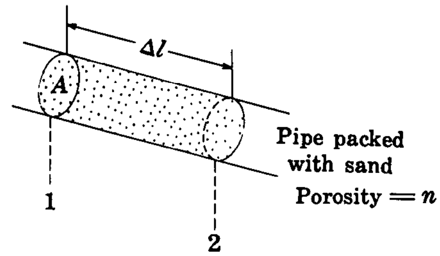
$$-\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right),$$

since $\Delta l \cdot n \cdot A$ gives the volume of the fluid element.

Our approach in this development is to equate the net force driving the flow to the frictional force opposing it; more exactly, we will obtain the vector sum of these opposing forces and set the result equal to zero. The resulting equation will be a statement of Darcy's law. We have obtained an expression for the net force driving the flow. We now consider the force opposing the motion. This force is due primarily to friction between the moving fluid and the porous medium. In some

## (19)—Con.

other systems of mechanics—for example in the case of a particle moving through a viscous fluid at moderate speed—the frictional retarding force is observed to be proportional to the velocity of movement. By analogy we assume a similar relation to hold for our element of fluid. However, as indicated in Part I, the actual (pore) velocity varies from point to point and is difficult or impossible to determine. For practical purposes therefore, we consider the frictional force on our fluid element to be proportional to the specific discharge, or flow per unit cross-sectional area, through the porous material. (See Section 14, Part I.) The specific discharge, which has the dimensions of a velocity (and is in fact a sort of apparent velocity), is determined by the statistical distribution of pore velocities within the fluid element; and we are, in effect, assuming that the total frictional retarding force on the element is likewise determined by this statistical distribution of pore velocities. In addition, we assume the total frictional retarding force on the fluid element to be proportional to the volume of fluid in the element, on the theory that the total area of fluid-solid contact within the element, and therefore the total frictional drag on the element, increases in proportion to the volume of the element. Finally, we assume that the retarding force is proportional to the dynamic viscosity of the fluid, since we would expect a fluid of low viscosity to move through a porous medium more readily than a highly viscous liquid.




QUESTION

Following the various assumptions outlined above, which of the following expressions would you choose as best representing the frictional retarding force on the fluid element of Section 1. (Shown again in the diagram.)

| | Turn to Section: |
|---|---|
| $-\frac{1}{k}\mu Q(\Delta l \cdot n \cdot A)$ | 2 |
| $-\frac{1}{k} \cdot \frac{Q^2\mu}{\Delta l \cdot n \cdot A}$ | 34 |
| $-\frac{1}{k}\mu(\Delta l \cdot n \cdot A)\frac{Q}{A}$ | 20 |

where $1/k$ indicates a constant of proportionality, $\mu$ is the dynamic viscosity of the fluid, and $Q$ is the fluid discharge through the pipe.

## (20)

Your answer in Section 19,

$$-\frac{1}{k}\mu(\Delta l \cdot n \cdot A)\frac{Q}{A},$$

is correct. The negative sign is employed to indicate that the frictional retarding force will be opposite in direction to the fluid movement. We assume that our fluid motion is steady—that is, that the fluid velocity is not changing with time, or in other words, that there is no fluid acceleration. In this condition, the forces producing the motion must be in balance with the frictional retarding force. The vector sum of these forces must therefore be zero; and because the force components contributing to the motion are all directed along the pipe, this vector sum is simply an algebraic sum.

## (20)—Con.

QUESTION

We have seen that the net driving force on the fluid element—that is, the net force in the direction of flow due to pressure and gravity together—is given by

$$-\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right)\Delta l\cdot n\cdot A.$$

Suppose we take the algebraic sum of this force and our retarding force, and set the result equal to zero. Which the following equations may then be derived from the result?

Turn to Section:

$\frac{dp}{dl}+\rho g\frac{dz}{dl}+\frac{\mu}{k}\cdot\frac{Q}{A}=\Delta l\cdot n\cdot A$ — 36

$-\frac{k}{\mu}\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right)=\frac{Q}{A}$ — 28

$\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right)\Delta l\cdot n\cdot A=\frac{\mu}{k}\frac{Q}{A}$ — 27

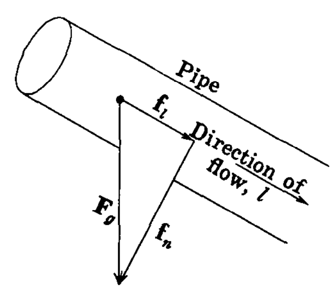
## (21)

Your answer in Section 7 is correct. Darcy's law, as an equation containing a derivative, is actually a differential equation. It relates flow per unit area, or flux, to the energy consumed per unit distance by friction. Analogies can readily be recognized between Darcy's law and the differential equations governing the steady flow of heat or electricity. The hydraulic conductivity, $K$, is analogous to thermal or electrical conductivity; while hydraulic head, $h$, is a potential analogous to temperature or voltage. (To be more correct, the term $Kh$ constitutes a ground-water velocity potential—that is, a function whose derivative yields the flow velocity—provided both the fluid and the porous medium are homogeneous and the medium is isotropic.)

**This concludes the programed instruction of Part II.** A discussion in text format dealing with generalizations of Darcy's law begins on the page following Section 37.

## (22)

Your answer in Section 15 is not correct. The diagram shows the gravitational force vector, $\mathbf{F}_g$, resolved into two components, one parallel to the direction of flow, $\mathbf{f}_l$, and one perpendicular to it, $\mathbf{f}_n$. If the flow were vertically downward,—that is, colinear with $\mathbf{F}_g$—the entire gravitational force would be effective in producing flow. In the situation shown, however, one component of the gravitational force—$\mathbf{f}_n$, or that perpendicular to the flow—is balanced by static forces exerted by the walls of the pipe. To view this in another way, we may note that the fluid velocity itself is a vector, in the direction $l$. No vector can have a component perpendicular to its own direction; so the velocity vector




has no component in the direction of $\mathbf{f}_n$. The force component $\mathbf{f}_n$ can therefore contribute nothing to the fluid velocity.

**Return to Section 15 and choose another answer.**

## (23)

Your answer in Section 8 is not correct. The pressure at a point in a fluid is a scalar quantity; it is not directional in character, and we say that it "acts in all directions." However, if we choose any small cross-sectional area within the fluid, we can measure a force against this area attributable to the pressure, regardless of the orientation of the area. This force is a vector, or directed quantity; it acts in a direction normal to the small area and has a magnitude equal to the product of the pressure and the area. In the example of Sections 1 and 8, we consider the pressure at two points, the upstream and downstream faces of our fluid element. At the upstream face we write an expression $p_1nA$ for the magnitude of the force in the direction of the flow. At the downstream face we are interested in a force opposing the flow—that is, acting in a direction opposite to the flow. The magnitude of this force is again given as a product of pressure, porosity, and face area, $p_2nA$; but because we are interested in the force acting *against* the flow, or in a direction opposite to that originally taken, we now introduce a negative sign. The *net* force on the fluid element along the axis of the pipe can now be obtained by algebraic addition of the two force expressions.

**Return to Section 8 and choose another answer.**

## (24)

Your answer in Section 11 is not correct. The idea here is simply to combine the expressions obtained for the net pressure force (see Section 26) and for the component of the gravitational force parallel to the pipe (see Section 11). Forces are always combined by means of vector addition. In this case, however, the two vectors we are considering are oriented in the same direction. That is, both the net pressure force and our component of the gravitational force are oriented in the direction of the flow. In this case, therefore, vector addition amounts to no more than the simple scalar addition of the magnitudes of the two components.

**Return to Section 11 and choose another answer.**

## (25)

Your answer in Section 1 is not correct. If we were dealing with open flow in the pipe, the force on the fluid element would indeed be given by the term $p_1A$. Here, however, a part of the area $A$ is occupied by solid sand grains and the remainder by the upstream face of the fluid element. For our purposes here, we may assume that the ratio of fluid area to total area is equal to the porosity, $n$.

**Return to Section 1 and choose another answer.**

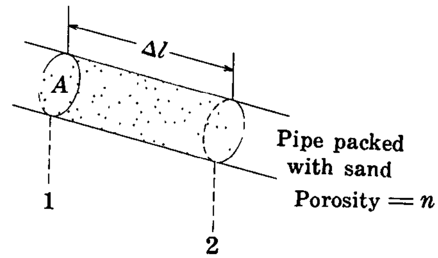
## (26)

Your answer in Section 31,

$$-\frac{dp}{dl}\Delta l n A,$$

is correct. The gradient or derivative of pressure, $dp/dl$, multiplied by the length interval, $\Delta l$, gives the change in pressure, $p_2-p_1$, occurring in that interval. Since we require the term $p_1-p_2$, we use a negative sign. Multiplication by the fluid area, $nA$, then gives the net pressure force on the element.

Our purpose in this chapter is to develop Darcy's law by equating the forces driving a flow to the frictional force retarding it. We have considered the pressure force, which is one of the forces driving the flow. In addition to this pressure force, the element of fluid is acted upon directly by the force of gravity. The total gravitational force on the element is given by the acceleration due to gravity, $g$, multiplied by the mass, $m$, of fluid in the element.




QUESTION

Which of the following equations for the mass of fluid in our element, which is shown again in the diagram, is correct?

| | Turn to Section: |
|---|---|
| $m=\Delta l\cdot n\cdot A$ | 3 |
| $m=\rho\cdot\Delta l\cdot A$ | 17 |
| $m=\rho\cdot\Delta l\cdot n\cdot A$ | 15 |

## (27)

Your answer,

$$\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right)\Delta l\cdot n\cdot A=\frac{\mu}{k}\frac{Q}{A}$$

in Section 20 is not correct. Each of the force terms—the net driving force and the retarding force—contains the expression $\Delta l\cdot n\cdot A$ representing the volume of fluid in the element. When these force terms are added and their sum set equal to zero, the term $\Delta l\cdot n\cdot A$ may be divided out of the equation.

**Return to Section 20 and choose another answer.**

## (28)

Your answer in Section 20,

$$-\frac{k}{\mu}\left(\frac{dp}{dl}+\rho g\frac{dz}{dl}\right)=\frac{Q}{A},$$

is correct. For the case of a fluid of uniform density and viscosity, the terms $\mu$ and $\rho$ are constants and may be combined with the other constants in the problem to form a new constant, $K$, defined as

$$K=\frac{k\rho g}{\mu}.$$

Using this new constant we may rewrite our equation in the form

$$-K\left(\frac{1}{\rho g}\frac{dp}{dl}+\frac{dz}{dl}\right)=\frac{Q}{A}.$$

(*continued on next page*)

## (28)—Con.

**QUESTION**

Keeping in mind that the term $1/\rho g$ is a constant, so that

$$\frac{1}{\rho g}\frac{dp}{dl}=\frac{d\left(\frac{p}{\rho g}\right)}{dl},$$

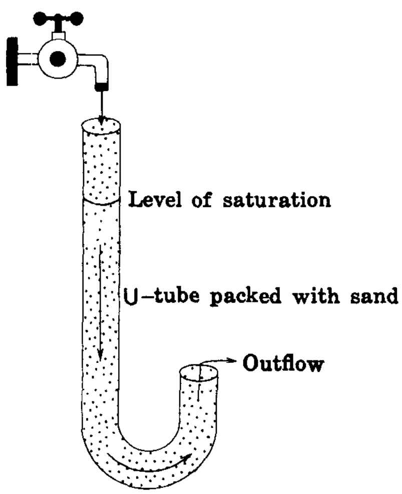
which of the equations given below constitutes a valid expression of the equation we have just obtained?

| | Turn to Section: |
|---|---|
| $\frac{Q}{A}=-K\frac{dh}{dl}$ | 7 |
| $\frac{Q}{A}=-K\left\{\frac{dp}{dl}+\frac{dz}{dl}\right\}$ | 9 |
| $\frac{Q}{A}=-K\left\{\frac{1}{\rho g}\frac{dp}{dl}+\frac{dh}{dl}\right\}$ | 30 |

$h$ represents the head as defined in Part I—that is,

$$h=\frac{p}{\rho g}+z.$$

## (29)




Your answer in Section 7 is not correct. Ground water frequently discharges upward into stream valleys; and in the figure, upward flow occurs in the shorter arm of the U-tube. Thus statement (a) of Section 7 cannot always be true.

**Return to Section 7 and choose another answer.**

## (30)

Your answer in Section 28 is not correct. We saw in Part I that hydraulic head, $h$, was given by

$$h=\frac{p}{\rho g}+z.$$

The derivative of $h$ with respect to distance, $l$, is therefore given by

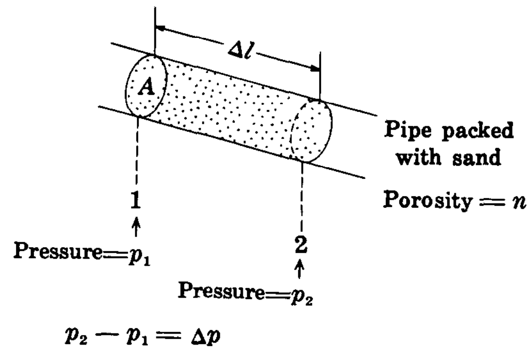
$$\frac{dh}{dl}=\frac{d\left\{\frac{p}{\rho g}+z\right\}}{dl}.$$

**Using this relation, return to Section 28 and choose another answer.**

## (31)

Your answer in Section 8 is correct. The net force in the direction of flow is given by the difference between the two opposing forces exerted upon the opposite faces of the element by the adjacent elements of fluid. We may now factor out the common term $nA$ and obtain as our expression for net pressure force $(p_1 - p_2)nA$, or $-\Delta p nA$, where $\Delta p$ indicates the small pressure difference, $p_2 - p_1$, between the downstream face of the fluid element and the upstream face.

Since pressure is varying from point to point within our system, we may speak of a pressure gradient; that is, a rate of change of pressure with distance, $l$, along the flow path. This gradient might be expressed, for example, in pounds per square inch (of pressure) per foot (of distance); it is represented by the symbol $dp/dl$, and is referred to as the derivative of pressure with respect to distance in the direction $l$. If we were to plot a graph of pressure versus distance, $dp/dl$ would represent the slope of the graph.




QUESTION

Which of the following expressions is approximately equivalent to the net pressure force, $-\Delta p nA$, on our element of fluid (shown again in the diagram)?

| | Turn to Section: |
|---|---|
| $\frac{dp}{dl}\Delta l nA$ | 26 |
| $\frac{dp}{dl}nA$ | 5 |
| $\Delta p \frac{dp}{dl} nA$ | 14 |

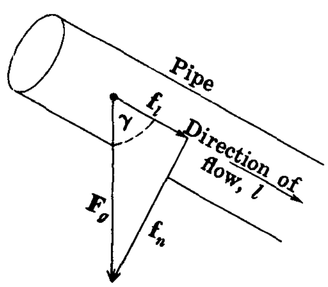
## (32)

Your answer, $\rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \sin\gamma$, in Section 35 is not correct. We have already seen that the magnitude of our force component is given by $\rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \cos\gamma$. In the answer you have chosen, $\sin\gamma$ has been substituted for $\cos\gamma$ in our original expression—and this can be true only for a particular value of the angle $\gamma$. It is true, however, that the idea of this question is to find an equivalent term for $\cos\gamma$ and substitute it in our previous expression for the force component.

**Return to Section 35 and choose another answer.**

## (33)

Your answer in Section 15 is correct; we may resolve the gravitational force, $\mathbf{F}_g$ into two orthogonal components, $\mathbf{f}_l$ and $\mathbf{f}_n$, parallel to and perpendicular to the axis of the pipe as shown in the figure. There is no movement perpendicular to the pipe; the component of the gravitational force in this direction is balanced by static forces exerted against the fluid element by the wall of the pipe. The component parallel to the pipe does contribute to the motion and must be taken into account in equations describing the flow.




###### QUESTION

The magnitude of the total gravitational force upon the element is given by the mass of the element multiplied by the acceleration due to gravity; that is, $\mathbf{F}_g = mg$, where $m$ is the mass of the fluid element. Referring to the diagram shown, which of the following expressions gives the magnitude of the component of the gravitational force parallel to the axis of the pipe?

| | Turn to Section: |
|---|---|
| $\mathbf{f}_l = \rho \cdot n \cdot \Delta l \cdot A \cdot g$ | 6 |
| $\mathbf{f}_l = \rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \cos \gamma$ | 35 |
| $\mathbf{f}_l = n \cdot \Delta l \cdot A \cdot g \cdot \tan \gamma$ | 37 |

## (34)

Your answer in Section 19,

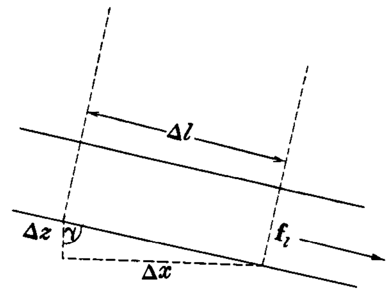
$$-\frac{1}{k}\frac{Q^2\mu}{\Delta l \cdot n \cdot A},$$

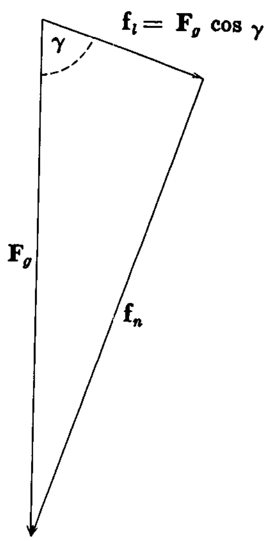
is not correct. Our assumptions were that the retarding force would be proportional in some way to the dynamic viscosity ($\mu$), to the volume of fluid in the element ($\Delta l \cdot n \cdot A$), and to the specific discharge, or flow per unit area ($Q/A$). Your answer represents the retarding force as proportional to the square of fluid discharge, which might be compatible with the assumptions, but as inversely proportional to the volume of fluid in the element, which is not compatible with the assumptions.

**Return to Section 19 and choose another answer.**

## (35)

Your answer, $\rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \cos \gamma$, in Section 33 is correct. The mass of the fluid element, as we have seen, is $\rho \cdot n \cdot \Delta l \cdot A$; multiplication by the acceleration, $g$, gives the total gravitational force on the element. The component of this force parallel to the pipe, as indicated by the vector diagram, will be found by multiplying the total force by the cosine of $\gamma$.




**QUESTION**

Suppose we now draw a small right triangle, taking the hypotenuse as $\Delta l$, the length of our fluid element, and constructing the two sides $\Delta z$ and $\Delta x$ as in the diagram. Which of the following expressions may then be used for the magnitude (without regard to sign) of the component of gravitational force parallel to the flow?

| | Turn to Section: |
|---|---|
| $\rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \sin \gamma$ | 32 |
| $\rho \cdot n \cdot A \cdot g \cdot \sqrt{(\Delta x)^2 + (\Delta z)^2}$ | 4 |
| $\rho \cdot n \cdot \Delta l \cdot A \cdot g \cdot \dfrac{\Delta z}{\Delta l}$ | 11 |

## (36)

Your answer in Section 20 is not correct. If the sum of the two force expressions is set equal to zero, we have

$$-\left(\frac{dp}{dl} + \rho g \frac{dz}{dl}\right)(\Delta l \cdot n \cdot A) - \frac{1}{k}\mu(\Delta l \cdot n \cdot A)\frac{Q}{A} = 0.$$

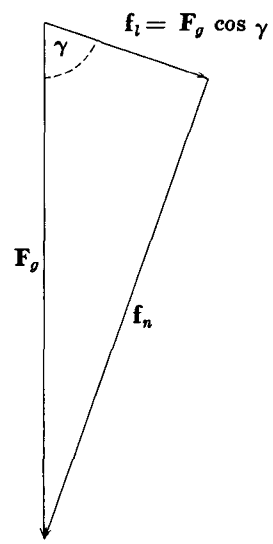
We may divide through by the term $\Delta l \cdot n \cdot A$, representing the volume of fluid in the element, and rearrange the resulting equation to obtain the required result.

**Return to Section 20 and choose another answer.**

## (37)

Your answer in Section 33 is not correct. The total gravitational force on the element is given by $mg$, where $m$ is the mass of fluid in the element and $g$ is the acceleration due to gravity. The mass of fluid in the element is in turn given by the volume of fluid in the element multiplied by the mass per unit volume, or mass density, of the fluid, which we have designated $\rho$. The volume of fluid in the element, as we have seen is $n \cdot \Delta l \cdot A$, where $n$ is the porosity. The mass is therefore $\rho \cdot n \cdot \Delta l \cdot A$; and the total force of gravity on the fluid element is given by

$$\mathbf{F}_g = \rho \cdot n \cdot \Delta l \cdot A \cdot g.$$




We require the component of this gravitational force parallel to the axis of the pipe. The sketch shows a vector diagram in which the length of each arrow is proportional to the force or component it represents. The gravitational force is represented by the arrow $\mathbf{F}_g$ and the components are represented by the arrows $\mathbf{f}_l$ and $\mathbf{f}_n$. The rule for the resolution of a vector into components can be visualized from geometric considerations. The magnitude of the component of a vector in a given direction is the product of the magnitude of the vector and the cosine of the angle between the direction of the vector and the given direction.

**Return to Section 33 and choose another answer.**

# Generalizations of Darcy's Law

The form of Darcy's law considered in the preceding program is useful only for one-dimensional flow. The discussion in this section indicates, in general outline, the manner in which Darcy's law is extended to cover more complex situations. Vector notation is used for economy of presentation, and this discussion is intended primarily for readers familiar with this notation. Those concepts which are essential to material covered later in the program are treated again as they are required in the development—without the use of vector notation. The material presented here is not difficult, and readers not familiar with vector notation may find it possible to follow the mathematics by reference to a standard text on vector analysis. However, those who prefer may simply read through this section for familiarity with qualitative aspects of the material and may then proceed directly to Part III.

For three-dimensional flow, we may consider the specific discharge, $q$ or $Q/A$, to be a vector quantity, with components $\mathbf{i}q_x$, $\mathbf{j}q_y$, and $\mathbf{k}q_z$ in the three coordinate directions. $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ represent the standard unit vectors of the Cartesian system. We consider a small area, $A_x$, oriented at right angles to the $x$ axis at a point 0, and observe the fluid discharge through this area to be $Q_x$; the limiting value of the ratio $Q_x/A_x$, as $A_x$ is made to shrink toward the point 0, gives the value of $q_x$ applicable at point 0. $q_y$ and $q_z$ are similarly defined for the $y$ and $z$ directions. The specific discharge at point 0 is given by the vector sum

$$q = \frac{Q}{A} = \mathbf{i}q_x + \mathbf{j}q_y + \mathbf{k}q_z.$$

$q$ is thus a vector point function; its magnitude and direction may vary with location in steady flow and with location and time in unsteady flow.

If the porous medium is homogeneous and isotropic and if the fluid is of uniform density and viscosity, the components of the specific-discharge vector are each given by a form of Darcy's law, utilizing the partial derivative of head with respect to distance in the direction in question. That is, the components are given by

$$q_x = -K\frac{\partial h}{\partial x}$$

$$q_y = -K\frac{\partial h}{\partial y}$$

$$q_z = -K\frac{\partial h}{\partial z}$$

where $K$ is the hydraulic conductivity.

It follows that the specific-discharge vector in this case will be given by

$$q = -K\left\{\mathbf{i}\frac{\partial h}{\partial x} + \mathbf{j}\frac{\partial h}{\partial y} + \mathbf{k}\frac{\partial h}{\partial z}\right\}$$

or

$$q = -K \nabla h$$

where $\nabla h$ denotes the head-gradient vector.

Thus, if the medium is isotropic and homogeneous, $-Kh$ constitutes a velocity potential; and the various methods of potential theory, as applied in studying heat flow and electricity, may be utilized in studying the ground-water motion. Since the specific-discharge vector is colinear with $\nabla h$, it will be oriented at right angles to the surfaces of equal head, and flownet analysis immediately suggests itself as a useful method of solving field problems.

In practice, one does not usually find homogeneous and isotropic aquifers with which to work; frequently, however, simply for lack of more detailed data, aquifers are assumed to be homogeneous and isotropic in obtaining initial or approximate solutions to ground-water problems.

The situation in many aquifers can be represented more successfully by a slightly more general form of Darcy's law, in which a different hydraulic conductivity is assigned to each of the coordinate directions. Darcy's law then takes the form

$$q_x = -K_x \frac{\partial h}{\partial x}$$

$$q_y = -K_y \frac{\partial h}{\partial y}$$

$$q_z = -K_z \frac{\partial h}{\partial z}$$

where $K_x$, $K_y$, and $K_z$ represent the hydraulic conductivities in the $x$, $y$, and $z$ directions, respectively, and again

$$q = \mathbf{i}q_x + \mathbf{j}q_y + \mathbf{k}q_z.$$

This form of Darcy's law can be applied only to those anisotropic aquifers which are characterized by three principal axes of hydraulic conductivity (or permeability) which are mutually orthogonal, so that the direction of maximum hydraulic conductivity is at right angles to the direction of minimum hydraulic conductivity. These axes must correspond with the $x$, $y$, and $z$ axes used in the analysis. The implication is that one of the principal axes of conductivity must be vertical; for unless the $z$ axis is taken in the vertical direction, the term $\partial h/\partial z$ cannot be used to represent the sum of the vertical pressure gradient and the gravitational force term.

It is easily demonstrated that the specific-discharge vector and the lines of flow are no longer orthogonal to the surfaces of equal head in this anistropic case, and that the conditions for the existence of a velocity potential are no longer satisfied. Formal mathematical solutions to field problems are essentially as easy to obtain as in the isotropic case, however, since a relatively simple transformation of scales can be introduced which converts the anisotropic system to an equivalent isotropic system (Muskat, 1937). The problem may then be solved in the equivalent isotropic system, and the solution retransformed to the original anisotropic system.

Probably the most common form of anisotropy encountered in the field is that exhibited by stratified sedimentary material, in which the permeability or hydraulic conductivity normal to the bedding is less than that parallel to the bedding. If the bedding is horizontal, the form of Darcy's law given above may be applied, using $K_x = K_y$. The anisotropy in this case is two-dimensional, with the axis of minimum permeability normal to the bedding, and the axis of maximum permeability parallel to it. In many cases, aquifers are assumed to exhibit simple two-dimensional anisotropy of this sort when in fact they are characterized by heterogeneous stratification and discrete alternations of permeability. This type of simplifying assumption frequently enables one to obtain an approximate solution, where otherwise no solution at all would be possible.

For many problems, however, this generalized form of Darcy's law is itself inadequate. As an example, one may consider a stratified aquifer, exhibiting simple two-dimensional anisotropy, which is not horizontal, but rather is dipping at an appreciable angle. The direction of minimum permeability, normal to the bedding, does not in this case coincide with the vertical. One may choose new coordinate axes to conform to the new principal directions of conductivity. If this is done, the component of the specific discharge in each of these new coordinate directions must be expressed in terms of the pressure gradient in the direction concerned, and the component of the gravitational force in that direction. Reduction of the equations to the simple form already given, using the principal directional derivatives of $h$, is not possible. Alternatively, one may retain the horizontal-vertical coordinate system, in which case the principal axes of conductivity do not coincide with the coordinate axes. In this case, hydraulic conductivity must be expressed as a tensor; the component of the specific discharge in one coordinate direction will not depend solely on the head gradient in that direction, but upon the head gradients in the other coordinate directions as well.

In addition to these considerations regarding aquifer anisotropy, practical problems require that attention be paid to heterogeneity, both of the aquifer and of the fluid. If the aquifer is heterogeneous, hydraulic conductivity must be treated as a function of the space coordinates; in this case, hydraulic conductivity (or in some cases intrinsic permeability) is usually defined as a tensor which varies with position in the aquifer.

If the fluid is heterogeneous, its viscosity and density cannot be treated as constants, as was done in the program section of Part II. Equations cannot be reduced to terms of the hydraulic conductivity and head gradients, but must rather be retained in terms of specific permeability, viscosity, pressure gradients, and components of the gravitational force (which depend upon fluid density, and will vary with position, and possibly with time, as fluid density varies). A special case of some importance is that in which the aquifer is horizontal, with principal axes of permeability in the $x$, $y$, and $z$ directions, but the fluid varies in both density and viscosity. Darcy's law for this case may be written

$$q_x = -\frac{k_x}{\mu_{x,y,z}} \frac{\partial p}{\partial x}$$

$$q_y = -\frac{k_y}{\mu_{x,y,z}} \frac{\partial p}{\partial y}$$

$$q_z = -\frac{k_z}{\mu_{x,y,z}} \left( \frac{\partial p}{\partial z} + \rho_{x,y,z} g \right)$$

and again

$$\mathbf{q} = \mathbf{i}q_x + \mathbf{j}q_y + \mathbf{k}q_z.$$

In these equations, $k_x$, $k_y$ and $k_z$ are the intrinsic permeabilities in the $x$, $y$, and $z$ directions; $\mu_{x,y,z}$ is the dynamic viscosity function; $\rho_{x,y,z}$ is the density function; and the other terms are as previously defined. Since gravity is assumed to have no components in the horizontal plane, density does not enter into the expressions for $q_x$ and $q_y$. In natural aquifers, variations in density are related primarily to variations in dissolved-solid content of the water, while variations in viscosity are related primarily to variations of ground-water temperature. The equations given above thus have utility in situations where water quality and water temperature are known to vary in an aquifer.